The Simplex Method

Linear Programming – Foundations and Algorithm

English

Wolfgang Boehme

wolfgang.boehme@economic-modeling.de

www.economic-modeling.de

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This document provides a systematic introduction to the Simplex method. It is intended for students with no prior knowledge of optimization.

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Table of Contents

1 Introduction and Motivation **2**

2 Fundamental Concepts of Linear Programming **2**
- 2.1 The General Linear Program . . . 2
- 2.2 Key Definitions . . . 3

3 Geometric Interpretation (2 Variables) **3**
- 3.1 Example: Geometric Solution . . . 3
- 3.2 Fundamental Theorem of Linear Programming . . . 4

4 Standard Form and Slack Variables **5**
- 4.1 Converting to Standard Form . . . 5
- 4.2 Basic Solutions and Basic Variables . . . 5

5 The Simplex Algorithm **6**
- 5.1 The Simplex Tableau . . . 6
- 5.2 Pivot Rules . . . 6
  - 5.2.1 Choosing the Pivot Column (Entering Variable) . . . 6
  - 5.2.2 Choosing the Pivot Row (Leaving Variable) . . . 7
  - 5.2.3 Pivot Operation . . . 7
- 5.3 Complete Algorithm . . . 7

6 Complete Example: 2 Variables **8**

7 Extended Example: 5 Variables, 4 Constraints **9**

8 Duality **9**
- 8.1 The Dual Linear Program . . . 9
- 8.2 Weak and Strong Duality Theorems . . . 10
- 8.3 Complementary Slackness . . . 10

9 Shadow Prices **11**
- 9.1 Economic Interpretation of Dual Variables . . . 11
- 9.2 Validity Range of Shadow Prices . . . 11

10 Dual Simplex Algorithm **12**
- 10.1 Motivation . . . 12
- 10.2 Algorithm . . . 12
- 10.3 Complete Example: Dual Simplex . . . 13

11 Special Cases **14**
- 11.1 Unbounded Problem . . . 14
- 11.2 Infeasible Problem . . . 14
- 11.3 Degeneracy . . . 14

12 Complexity and Practical Remarks **14**
12.1 Theoretical Complexity 14
12.2 Duality 15
12.3 Practical Remarks 15

13 Summary **16**

1 Introduction and Motivation

Many decision problems can be formulated mathematically as **linear optimization problems**: we seek the best solution – measured by a linear objective function – subject to certain linear inequalities and equations.

Typical Applications

- **Production:** What quantities of different products should be manufactured to maximize profit, subject to capacity constraints?
- **Logistics:** How can transportation routes be planned to minimize costs?
- **Finance:** How can a portfolio be constructed to maximize return for a given level of risk?
- **Diet planning:** How can foods be combined so that all nutritional requirements are met at minimum cost?

The **Simplex method** (*Simplex Algorithm*), developed by George B. Dantzig in 1947, is the classical procedure for solving such problems. To this day, it remains one of the most widely used optimization methods in practice.

2 Fundamental Concepts of Linear Programming

2.1 The General Linear Program

Definition: Linear Program (LP)

A **linear program** (LP) has the form:

$$\begin{aligned} &\text{Maximize (or Minimize)} && z = c_1x_1 + c_2x_2 + \cdots + c_nx_n && && (1)\\ &\text{subject to} && a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \;\leq\; b_i, && i = 1,\ldots,m && (2)\\ & && x_j \geq 0, && j = 1,\ldots,n && (3) \end{aligned}$$

where $c_j, a_{ij}, b_i \in \mathbb{R}$ are given constants and $x_j \geq 0$ denote the **decision variables**.

In matrix notation the LP reads:

$$\max\ \mathbf{c}^\top \mathbf{x} \quad \text{unter} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}$$

mit $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$.

2.2 Key Definitions

Definition: Feasible Region and Optimal Solution

- An assignment $(x_1, \ldots, x_n)$ is called **feasible** if it satisfies all constraints (2) and non-negativity conditions (3).
- The set of all feasible assignments is called the **feasible region** $\mathcal{F}$.
- A feasible assignment $\mathbf{x}^*$ is an **optimal solution** if $\mathbf{c}^\top \mathbf{x}^* \geq \mathbf{c}^\top \mathbf{x}$ for all $\mathbf{x} \in \mathcal{F}$ (maximization).

Definition: Convex Set and Vertex

- A set $\mathcal{F} \subseteq \mathbb{R}^n$ is **convex** if for all $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ and $\lambda \in [0,1]$: $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in \mathcal{F}$.
- A point $\mathbf{x} \in \mathcal{F}$ is a **vertex** (or extreme point) if it cannot be expressed as a proper convex combination of two other points in $\mathcal{F}$.

The feasible region of an LP is always a **convex polytope** – the intersection of finitely many half-spaces.

3 Geometric Interpretation (2 Variables)

Für den Fall $n = 2$ lässt sich das LP anschaulich in der Ebene darstellen.

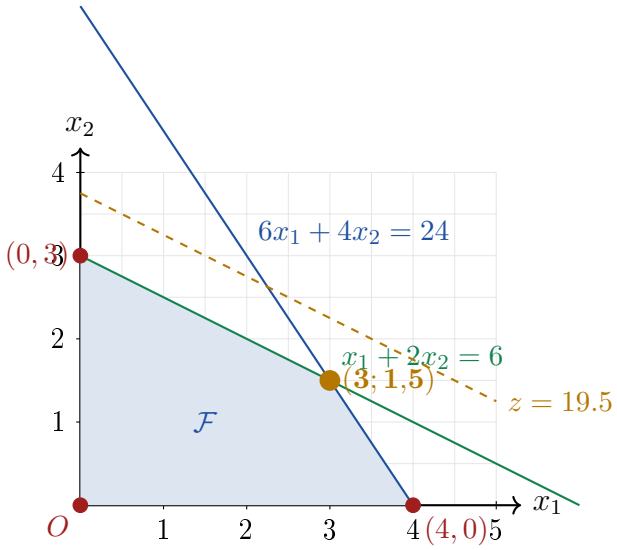
3.1 Example: Geometric Solution

Example 1: Production Planning

A company produces two products P_1 and P_2. The contribution margin is 5 units per unit of P_1 and 4 units per unit of P_2. The following capacity constraints apply:

$$\max \; z = 5x_1 + 4x_2$$

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 \quad \text{(machine capacity)} \\ x_1 + 2x_2 &\leq 6 \quad \text{(storage capacity)} \\ x_1, x_2 &\geq 0 \end{aligned}$$



The vertices of the feasible region $\mathcal{F}$ are $O = (0, 0)$, $(4, 0)$, $(3, 1.5)$ and $(0, 3)$. The optimum is at vertex $(3, 1.5)$ with $z^* = 5 \cdot 3 + 4 \cdot 1.5 = 19.5$.

3.2 Fundamental Theorem of Linear Programming

Theorem 1: Fundamental Theorem of LP

Let the LP be feasible and bounded. Then:

1. The feasible region $\mathcal{F}$ is a convex set.
2. An optimal solution always exists at a vertex of $\mathcal{F}$.
3. If $\mathcal{F}$ is non-empty and bounded, the LP has a finite optimal solution.

Proof. **(1):** Each constraint $\mathbf{a}_i^\top \mathbf{x} \leq b_i$ defines a convex half-space. The intersection of convex sets is convex.

(2): Let $\mathbf{x}^*$ be optimal but not an extreme point. Then there exist $\mathbf{u}, \mathbf{v} \in \mathcal{F}$ with $\mathbf{u} \neq \mathbf{v}$ and $\lambda \in (0, 1)$ such that $\mathbf{x}^* = \lambda\mathbf{u} + (1 - \lambda)\mathbf{v}$. By linearity of the objective:

$$\mathbf{c}^\top \mathbf{x}^* = \lambda \mathbf{c}^\top \mathbf{u} + (1 - \lambda)\mathbf{c}^\top \mathbf{v}$$

Hence at least one of $\mathbf{c}^\top \mathbf{u}$ or $\mathbf{c}^\top \mathbf{v}$ is at least as large as $\mathbf{c}^\top \mathbf{x}^*$ – so an equally good solution exists at an extreme point. Repeating this argument always leads to a vertex.

(3): Since $\mathcal{F}$ is non-empty and, as a convex polytope, has only finitely many extreme points, the maximum of the linear objective over $\mathcal{F}$ is always attained. □

Implication for the algorithm: Since the optimum always lies at a vertex and a convex polytope has only *finitely many* vertices, it suffices to move systematically from vertex to vertex while improving the objective. This is precisely the core idea of the Simplex method.

4 Standard Form and Slack Variables

4.1 Converting to Standard Form

The Simplex method operates on the **standard form** of the LP, in which all constraints appear as equalities.

Definition: Standard Form

An LP in standard form reads:

$$\max \ \mathbf{c}^\top \mathbf{x} \quad \text{unter} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}$$

mit $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \geq \mathbf{0}$.

Converting an inequality to an equality:
For each $\leq$-constraint $\mathbf{a}_i^\top \mathbf{x} \leq b_i$ we introduce a **slack variable** $s_i \geq 0$:

$$\mathbf{a}_i^\top \mathbf{x} + s_i = b_i, \quad s_i \geq 0$$

The slack variable s_i measures how far the constraint is from its boundary.

Example 1 (continued): Introducing Slack Variables

$$\begin{aligned} 6x_1 + 4x_2 + s_1 &= 24 \\ x_1 + 2x_2 + s_2 &= 6 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

The LP now has $n + m = 2 + 2 = 4$ variables and $m = 2$ equations.

4.2 Basic Solutions and Basic Variables

Definition: Basis – Basic Variables – Basic Solution

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m \leq n$.

- A **basis** B is a selection of m linearly independent columns of $\mathbf{A}$.
- The corresponding variables are **basic variables** x_B; the remaining ones **non-basic variables** x_N.
- Setting $x_N = \mathbf{0}$ yields the **basic solution**: $x_B = \mathbf{A}_B^{-1}\mathbf{b}$.
- A basic solution with $x_B \geq \mathbf{0}$ is a **basic feasible solution** (BFS).

Theorem 2: Vertices and Basic Feasible Solutions

Every vertex of the feasible region $\mathcal{F}$ corresponds to a basic feasible solution, and vice versa.

Proof sketch. A point $\mathbf{x} \in \mathcal{F}$ is a vertex if and only if the active constraints (those with $\mathbf{a}_i^\top \mathbf{x} = b_i$) together with $x_j = 0$ for sufficiently many j form a unique system – which is precisely the definition of a basic solution. □

The **initial basic feasible solution** for our example is $x_1 = x_2 = 0$, $s_1 = 24$, $s_2 = 6$ (the origin).

5 The Simplex Algorithm

5.1 The Simplex Tableau

The Simplex tableau summarizes the LP compactly. It contains the coefficient matrix, the right-hand side, and the objective row.

Basis	x_1	x_2	s_1	s_2	b
s_1	6	4	1	0	24
s_2	1	2	0	1	6
z	-5	-4	0	0	0

Convention: The objective row contains the **reduced costs** $\bar{c}_j$. Negative values $\bar{c}_j < 0$ indicate the objective can still be improved.

5.2 Pivot Rules

Definition: Reduced Costs and Optimality Criterion

The **reduced costs** of non-basic variable x_j are:

$$\bar{c}_j = c_j - \mathbf{c}_B^\top \mathbf{A}_B^{-1} \mathbf{a}_j$$

A BFS is optimal if $\bar{c}_j \leq 0$ for all non-basic variables x_j (maximization).

5.2.1 Choosing the Pivot Column (Entering Variable)

Rule: Dantzig Rule (Largest Improvement per Variable)

Choose as the entering variable x_s the non-basic variable with the **most negative** reduced cost:

$$s = \arg\min_j \bar{c}_j$$

Rationale: Increasing x_s by one unit improves the objective by $|\bar{c}_s|$. The Dantzig rule selects the variable with the greatest improvement rate.

5.2.2 Choosing the Pivot Row (Leaving Variable)

Rule: Minimum Ratio Test

For the entering variable x_s compute the **minimum ratio**:

$$r = \arg\min_{i:\, a_{is} > 0} \frac{b_i}{a_{is}}$$

The basic variable in row r leaves the basis.

Proof: Feasibility Preservation. Increasing x_s from 0 to $\theta > 0$, the basic variables change as:

$$x_{B_i} = \bar{b}_i - a_{is} \cdot \theta, \quad i = 1, \ldots, m$$

For all $x_{B_i} \geq 0$ to remain satisfied:

$$\theta \leq \frac{\bar{b}_i}{a_{is}} \quad \text{für alle } i \text{ mit } a_{is} > 0$$

The maximum allowable value is $\theta^* = \min_{i:\, a_{is}>0} \frac{\bar{b}_i}{a_{is}}$. The row achieving this minimum becomes the pivot row – the basic variable there becomes 0 and leaves. □

5.2.3 Pivot Operation

Let (r, s) be the pivot element a_{rs}. The **pivot operation** transforms the tableau:

1. **Normalize pivot row:** Divide row r by a_{rs}:

$$\text{Zeile}_r \leftarrow \frac{1}{a_{rs}} \cdot \text{Zeile}_r$$

2. **Eliminate in all other rows:** For $i \neq r$:

$$\text{Zeile}_i \leftarrow \text{Zeile}_i - a_{is} \cdot \text{Zeile}_r$$

3. **Eliminate in the objective row:**

$$\text{Zeile}_z \leftarrow \text{Zeile}_z - \bar{c}_s \cdot \text{Zeile}_r$$

After the pivot, x_s is in the basis (unit vector in column s).

5.3 Complete Algorithm

Algorithm: Simplex Method

1. **Initialization:** Set up the initial tableau (slack variables as basic variables).
2. **Optimality test:** Are all reduced costs $\bar{c}_j \leq 0$? Then the current BFS is optimal. STOP.
3. **Pivot column:** Choose $s = \arg\min_j \bar{c}_j < 0$.

4. **Unboundedness test:** Is $a_{is} \leq 0$ for all i? Then the LP is unbounded. STOP.
5. **Pivot row:** Choose $r = \arg\min_{i:\, a_{is}>0} \bar{b}_i / a_{is}$.
6. **Pivot operation:** Perform the basis change (x_s enters, x_{B_r} leaves).
7. Go to Step 2.

6 Complete Example: 2 Variables

We solve Example 1 completely using the Simplex method.

Initial tableau:

Basis	x_1	x_2	s_1	s_2	b
s_1	6	4	1	0	24
s_2	1	2	0	1	6
z	-5	-4	0	0	0

Iteration 1: $\bar{c}_1 = -5$ is the most negative $\Rightarrow$ x_1 enters. Ratios: $24/6 = 4$, $6/1 = 6$. Minimum: $4 \Rightarrow$ s_1 leaves. Pivot element: **6**.

Pivot: Row 1 $\div 6$; then Row 2 $-1\cdot$ Row 1; Row z $+5\cdot$ Row 1.

Tableau after Iteration 1:

Basis	x_1	x_2	s_1	s_2	b
x_1	1	2/3	1/6	0	4
s_2	0	4/3	$-1/6$	1	2
z	0	$-2/3$	5/6	0	20

Iteration 2: $\bar{c}_2 = -2/3 < 0 \Rightarrow$ x_2 enters. Ratios: $4/(2/3) = 6$, $2/(4/3) = 3/2$. Minimum: $3/2 \Rightarrow$ s_2 leaves. Pivot: **4/3**.

Pivot: Row 2 $\cdot\frac{3}{4}$; then Row 1 $-\frac{2}{3}\cdot$ Row 2; Row z $+\frac{2}{3}\cdot$ Row 2.

Tableau after Iteration 2 (optimal):

Basis	x_1	x_2	s_1	s_2	b
x_1	1	0	1/4	$-1/2$	3
x_2	0	1	$-1/8$	3/4	3/2
z	0	0	3/4	1/2	**19,5**

All reduced costs ≥ 0: **Optimality reached.**

$$\boxed{x_1^* = 3, \quad x_2^* = 1{,}5, \quad z^* = 19{,}5}$$

7 Extended Example: 5 Variables, 4 Constraints

Example 2: Production Programme Planning

$$\max\ z = 5x_1 + 4x_2 + 3x_3 + 6x_4 + 2x_5$$

$$\begin{aligned} 6x_1 + 4x_2 + 2x_3 + 5x_4 + x_5 &\leq 240 \\ 3x_1 + 2x_2 + 5x_3 + 4x_4 + 3x_5 &\leq 270 \\ 5x_1 + 6x_2 + 3x_3 + 2x_4 + 4x_5 &\leq 420 \\ 2x_1 + x_2 + 4x_3 + 3x_4 + 2x_5 &\leq 350 \\ x_1, \ldots, x_5 &\geq 0 \end{aligned}$$

After introducing slack variables $s_1, \ldots, s_4$, we obtain an LP with 9 variables and 4 equations. The Simplex tool computes the optimal solution (for verification).

Note: With 5 variables and 4 constraints, the algorithm may visit up to $\binom{9}{4} = 126$ basic solutions. In practice, far fewer iterations suffice.

8 Duality

8.1 The Dual Linear Program

For every linear program – the **primal problem** – there exists an associated **dual problem**. Duality is one of the most fundamental concepts in linear programming.

Definition: Primal and Dual Problem

Let the **primal problem** be given as:

$$\text{(P)} \quad \max\ \mathbf{c}^\top \mathbf{x} \quad \text{unter} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}$$

Then the associated **dual problem** is:

$$\text{(D)} \quad \min\ \mathbf{b}^\top \mathbf{y} \quad \text{unter} \quad \mathbf{A}^\top \mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0}$$

The variables $y_i \geq 0$ are called **dual variables** or **shadow prices**.

Construction rules:

- The dual has as many variables y_i as the primal has constraints (m each).
- The dual has as many constraints as the primal has variables (n each).
- The coefficient matrix of the dual is the **transpose** $\mathbf{A}^\top$ of the primal.
- Primal max $\rightarrow$ dual min; primal $\leq \rightarrow$ dual $\geq$.

Example 1 (continued): Dual Problem

The primal reads:

$$\max\ z = 5x_1 + 4x_2 \quad \text{u.d.N.} \quad 6x_1 + 4x_2 \leq 24, \quad x_1 + 2x_2 \leq 6, \quad x_1, x_2 \geq 0$$

The associated dual:

$$\min\ w = 24y_1 + 6y_2 \quad \text{u.d.N.} \quad 6y_1 + y_2 \geq 5, \quad 4y_1 + 2y_2 \geq 4, \quad y_1, y_2 \geq 0$$

8.2 Weak and Strong Duality Theorems

Theorem 3: Weak Duality Theorem

Let $\mathbf{x}$ be a feasible solution of (P) and $\mathbf{y}$ a feasible solution of (D). Then:

$$\mathbf{c}^\top \mathbf{x} \leq \mathbf{b}^\top \mathbf{y}$$

Proof. Da $\mathbf{x} \geq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$, $\mathbf{Ax} \leq \mathbf{b}$ und $\mathbf{A}^\top \mathbf{y} \geq \mathbf{c}$:

$$\mathbf{c}^\top \mathbf{x} \leq (\mathbf{A}^\top \mathbf{y})^\top \mathbf{x} = \mathbf{y}^\top \mathbf{A}\mathbf{x} \leq \mathbf{y}^\top \mathbf{b} = \mathbf{b}^\top \mathbf{y}$$

□

Corollary: Every dual feasible solution provides an **upper bound** for the primal optimal value. The optimal primal value always lies in the interval $[\mathbf{c}^\top \mathbf{x}, \mathbf{b}^\top \mathbf{y}]$ for any feasible $\mathbf{x}$, $\mathbf{y}$.

Theorem 4: Strong Duality Theorem

If (P) has a finite optimal solution $\mathbf{x}^*$, then (D) also has a finite optimal solution $\mathbf{y}^*$, and:

$$\mathbf{c}^\top \mathbf{x}^* = \mathbf{b}^\top \mathbf{y}^*$$

Proof sketch. In the optimal Simplex tableau, the objective row under slack variable s_i contains exactly y_i^* (with correct sign). Since all reduced costs ≥ 0 and all right-hand sides ≥ 0, $\mathbf{y}^* = \mathbf{c}_B^\top \mathbf{A}_B^{-1}$ is both dual feasible and optimal. The equality $\mathbf{c}^\top \mathbf{x}^* = \mathbf{b}^\top \mathbf{y}^*$ follows from the complementary slackness conditions. □

8.3 Complementary Slackness

Theorem 5: Complementary Slackness Condition (CSC)

$\mathbf{x}^*$ and $\mathbf{y}^*$ are simultaneously optimal for (P) and (D) if and only if:

$$y_i^* \cdot (b_i - \mathbf{a}_i^\top \mathbf{x}^*) = 0 \quad \forall\, i = 1, \ldots, m \tag{4}$$

$$x_j^* \cdot (\mathbf{a}_j^\top \mathbf{y}^* - c_j) = 0 \quad \forall\, j = 1, \ldots, n \tag{5}$$

Interpretation:

- (4): If $y_i^* > 0$ (dual variable active), the i-th primal constraint must be **binding** ($b_i - \mathbf{a}_i^\top \mathbf{x}^* = 0$, no slack).
- (5): If $x_j^* > 0$ (primal variable active), the j-th dual constraint must be binding.

9 Shadow Prices

9.1 Economic Interpretation of Dual Variables

Definition: Shadow Price

The **shadow price** y_i^* of the i-th constraint indicates how much the optimal value z^* changes when the right-hand side b_i is increased by one unit – assuming the current optimal basis is maintained:

$$y_i^* = \frac{\partial z^*}{\partial b_i}$$

Derivation: In the optimal tableau $\mathbf{x}_B^* = \mathbf{A}_B^{-1}\mathbf{b}$. A change $\Delta\mathbf{b}$ leads to:

$$\Delta z^* = \mathbf{c}_B^\top \mathbf{A}_B^{-1}\Delta\mathbf{b} = (\mathbf{y}^*)^\top \Delta\mathbf{b} = \sum_{i=1}^{m} y_i^* \cdot \Delta b_i$$

Shadow prices can be read directly from the optimal tableau: they appear in the objective row under the slack variable columns s_i.

9.2 Validity Range of Shadow Prices

The linear relationship $\Delta z^* = \mathbf{y}^{*\top}\Delta\mathbf{b}$ holds only as long as the **optimal basis remains unchanged**. The validity range follows from the condition $\mathbf{A}_B^{-1}(\mathbf{b}+\Delta\mathbf{b}) \geq \mathbf{0}$:

$$\Delta b_i \in \left[-\frac{(\mathbf{A}_B^{-1}\mathbf{b})_k}{(\mathbf{A}_B^{-1}\mathbf{e}_i)_k}, \ \ldots \right] \quad \text{für alle } k \text{ mit } (\mathbf{A}_B^{-1}\mathbf{e}_i)_k > 0$$

Example 1 (continued): Shadow Prices

In the optimal tableau (after two iterations), the objective row under s_1 and s_2 contains:

$$y_1^* = \frac{3}{4}, \qquad y_2^* = \frac{1}{2}$$

Interpretation:

- One additional machine hour ($b_1 = 25$ instead of 24) increases z^* by 0.75 units.
- One additional storage unit ($b_2 = 7$ instead of 6) increases z^* by 0.5 units.
- Both constraints are **binding**: $s_1^* = s_2^* = 0$. Hence both shadow prices are positive.

Verification with the dual optimum:

$$w^* = 24 \cdot \tfrac{3}{4} + 6 \cdot \tfrac{1}{2} = 18 + 3 = 21 = z^* \checkmark$$

10 Dual Simplex Algorithm

10.1 Motivation

The primal Simplex always maintains **primal feasibility** (all $\bar{b}_i \geq 0$) and seeks **dual feasibility** (all $\bar{c}_j \geq 0$). The **dual Simplex** reverses this: it starts with a dually feasible but primal infeasible tableau and restores primal feasibility step by step.

Definition: Dually Feasible Tableau

A tableau is called **dually feasible** if all reduced costs are non-negative: $\bar{c}_j \geq 0$ for all j. However, it may be primal infeasible: $\bar{b}_i < 0$ for at least one i.

Typical applications:

- Problems with $\geq$-constraints (dually feasible after multiplication by -1)
- **Sensitivity analysis:** After a change in the right-hand side $\mathbf{b}$, the current optimal tableau may become primal infeasible – the dual Simplex re-optimises efficiently.
- Branch & Bound in integer programming

10.2 Algorithm

Algorithm: Dual Simplex

1. **Initialization:** Set up a dually feasible initial tableau ($\bar{c}_j \geq 0$ for all j).
2. **Primal feasibility test:** Is $\bar{b}_i \geq 0$ for all i? Then the solution is **primal and dual optimal**. STOP.
3. **Pivot row:** Choose $r = \arg\min_i \bar{b}_i < 0$ (most negative right-hand side).
4. **Infeasibility test:** Is $a_{rj} \geq 0$ for all j? Then the LP is **infeasible**. STOP.
5. **Pivot column:** Choose s by the **dual ratio test**:
$$s = \arg\min_{j:\, a_{rj}<0} \frac{\bar{c}_j}{|a_{rj}|}$$
6. **Pivot operation:** Perform the basis change.
7. Go to Step 2.

Proof: Dual feasibility is preserved. After the pivot operation with pivot element $a_{rs} < 0$, the reduced cost $\bar{c}_j$ changes to:

$$\bar{c}'_j = \bar{c}_j - \frac{a_{rj}}{a_{rs}} \cdot \bar{c}_s$$

Since $\bar{c}_s = 0$ (basic variable before pivot), $\bar{c}_j$ for non-basic variables $j \neq s$ changes as:

$$\bar{c}'_j = \bar{c}_j - \frac{a_{rj}}{a_{rs}} \cdot \bar{c}_s^{(\text{neu})}$$

Die Pivotspaltenregel $s = \arg\min_{a_{rj}<0} \bar{c}_j/|a_{rj}|$ garantiert $\bar{c}'_j \geq 0$ für alle j. □

10.3 Complete Example: Dual Simplex

Example 3: Dual Simplex

$$\max\ z = -2x_1 - 3x_2 \qquad (\text{äquivalent: } \min\ 2x_1 + 3x_2)$$

$$\begin{aligned} x_1 + x_2 \geq 4 \quad &\Rightarrow \quad -x_1 - x_2 \leq -4 \\ 2x_1 + x_2 \geq 6 \quad &\Rightarrow \quad -2x_1 - x_2 \leq -6 \\ x_1, x_2 \geq 0 \quad & \end{aligned}$$

Initial tableau (after reformulation; slack variables s_1, s_2):

Basis	x_1	x_2	s_1	s_2	b
s_1	-1	-1	1	0	-4
s_2	-2	-1	0	1	-6
z	-2	-3	0	0	0

Dually feasible? All $\bar{c}_j \leq 0$ in the z-row (note signs: $-(-2) = 2 \geq 0$, $-(-3) = 3 \geq 0$). ✓

Iteration 1: Pivot row: $r = 2$ ($\bar{b}_2 = -6$ is most negative). Dual ratios: $\bar{c}_1/|a_{21}| = 2/2 = 1$, $\bar{c}_2/|a_{22}| = 3/1 = 3$. Minimum: $s = 1$ (x_1 enters). Pivot element: $a_{21} = -2$.

Tableau after Iteration 1:

Basis	x_1	x_2	s_1	s_2	b
s_1	0	$-1/2$	1	$-1/2$	-1
x_1	1	$1/2$	0	$-1/2$	3
z	0	-2	0	-1	6

Iteration 2: Pivot row: $r = 1$ ($\bar{b}_1 = -1$). Dual ratios: $\bar{c}_2/|a_{12}| = 2/(1/2) = 4$, $\bar{c}_4/|a_{14}| = 1/(1/2) = 2$. Minimum: $s = 4$ (s_2 enters). Pivot element: $a_{14} = -1/2$.

Optimal tableau:

Basis	x_1	x_2	s_1	s_2	b
s_2	0	1	-2	1	2
x_1	1	1	-1	0	2
z	0	0	-2	0	**8**

All $\bar{b}_i \geq 0$: **Primal and dual optimal.**

$$\boxed{x_1^* = 2, \quad x_2^* = 0, \quad z^* = -8\ (\min\ 2x_1 + 3x_2 = 4 \Rightarrow z^*_{\min} = 4)}$$

11 Special Cases

11.1 Unbounded Problem

Definition: Unboundedness

An LP is **unbounded** if the objective can grow arbitrarily, i.e. $\sup_{\mathbf{x}\in\mathcal{F}} \mathbf{c}^\top \mathbf{x} = +\infty$.

Detection: In Step 4: all entries in the pivot column are ≤ 0 – no minimum ratio exists.

11.2 Infeasible Problem

Definition: Infeasibility

An LP is **infeasible** if the feasible region is empty: $\mathcal{F} = \emptyset$.

This occurs when constraints are contradictory (e.g. $x_1 \leq 2$ and $x_1 \geq 5$). For problems with $\geq$-constraints no obvious starting basis exists – the **two-phase method** or **Big-M method** is required.

11.3 Degeneracy

Definition: Degeneracy

A BFS is **degenerate** if a basic variable takes the value 0.

With degeneracy the minimum ratio test may yield 0, so the objective does not improve (**stalling**). In the worst case cycling may occur – prevented in practice by **Bland's rule** or the **lexicographic rule**.

Theorem: Convergence of Simplex (non-degenerate)

Assuming no degenerate BFS occur, the Simplex terminates in finitely many steps with an optimal solution or a proof of unboundedness.

Proof. Since $\mathcal{F}$ has only finitely many vertices and the objective *strictly* increases at each step, no vertex is visited twice. The algorithm terminates in at most $\binom{n}{m}$ steps. □

12 Complexity and Practical Remarks

12.1 Theoretical Complexity

The number of vertices can be exponential in n and m in the worst case (Klee–Minty, 1972). Nevertheless, the Simplex method is very efficient in practice:

- Typical iterations: $O(m)$ to $O(3m)$

- **Polynomial algorithms:** The ellipsoid method (Khachiyan, 1979) and interior-point method (Karmarkar, 1984) are theoretically polynomial but slower for many practical problems.

12.2 Duality

For every LP (primal) there exists a **dual LP**. The **strong duality theorem** states: if both are feasible, their optimal values coincide. The optimal Simplex tableau contains the dual variables directly in the objective row (shadow prices).

12.3 Practical Remarks

- For very large problems ($n, m > 10^4$), the **Revised Simplex** or **interior-point methods** are used.
- **Software:** CPLEX, Gurobi, GLPK, SciPy (Python), linprog (MATLAB).
- For **integer** variables, Simplex alone is insufficient – Branch & Bound is required (see next chapter).

13 Summary

Key Points of the Simplex Method

1. The **optimum always lies at a vertex** of the feasible region (fundamental theorem).
2. Vertices correspond to **basic feasible solutions** – m variables in the basis, $n - m$ at 0.
3. The Simplex moves from vertex to vertex while **always improving the objective**.
4. The **pivot column** is determined by the most negative reduced cost.
5. The **pivot row** is determined by the minimum ratio test (feasibility).
6. **Optimality**: all reduced costs ≥ 0.
7. **Unboundedness**: pivot column has no positive entry.
8. Termination is guaranteed when no **degeneracy** occurs.
9. Every LP has a **dual LP**. At the optimum $z^* = w^*$ (strong duality theorem).
10. **Shadow prices** y_i^* measure the value of one additional unit of the i-th resource.
11. The **dual Simplex** starts dually feasible and restores primal feasibility step by step.
12. **Complementary slackness:** Binding primal constraint $\Leftrightarrow$ positive shadow price.