Exercise Collection

Multi-Objective Optimization – Pareto Front and Solution Methods

English

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1 Level 1: Easy

Exercise 1.1 – Dominance Check and Pareto Set

Exercise 1.1

Six solutions are given with two objective functions (both to be minimized):

Solution	z_1	z_2	Pareto-optimal?
A	2	8	?
B	4	5	?
C	6	3	?
D	3	7	?
E	5	5	?
F	1	9	?

Tasks:

(a) Determine for each pair whether one solution dominates the other.

(b) Identify the Pareto-optimal solutions.

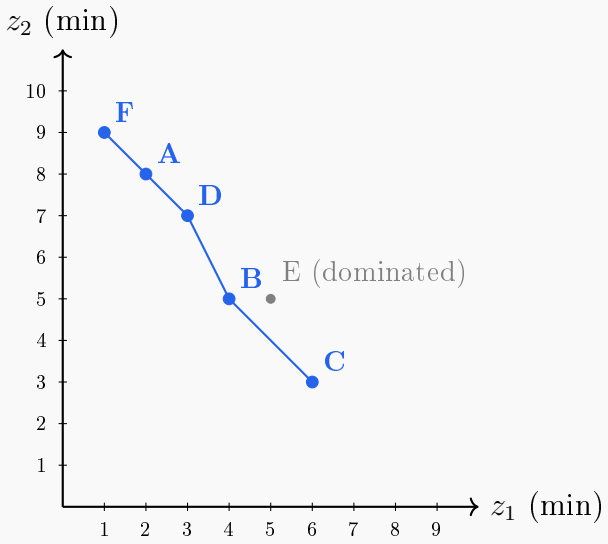
(c) Plot the solutions on a (z_1, z_2) diagram and mark the Pareto front.

(d) Determine the ideal and nadir point of the Pareto front.

Solution: Exercise 1.1

(a) Dominance check:
Does D dominate A? $z_1(D) = 3 > z_1(A) = 2$ – no (D is worse in z_1).
Does A dominate D? $z_1(A) = 2 \leq 3$, $z_2(A) = 8 > 7$ – no (A is worse in z_2).
Does B dominate E? $z_1(B) = 4 \leq 5$, $z_2(B) = 5 \leq 5$ with z_1 strictly smaller – **yes, B dominates E.**
Does B dominate D? $z_1(B) = 4 > 3$ – no.
Does C dominate B? $z_1(C) = 6 > 4$ – no.
Does A dominate F? $z_1(A) = 2 > 1$ – no. Does F dominate A? $z_1(F) = 1 < 2$, $z_2(F) = 9 > 8$ – no (F is worse in z_2).
(b) Pareto-optimal solutions:
E is dominated by B $\Rightarrow$ E is **not** Pareto-optimal. All others (A, B, C, D, F) are dominated by no other solution $\Rightarrow$ **Pareto set: {A, B, C, D, F}**.

(c) Diagram:



(d) Ideal and nadir point:

$$\mathbf{z}^{\text{id}} = (\min z_1, \min z_2) = (1, 3), \qquad \mathbf{z}^{\text{nad}} = (\max z_1, \max z_2)|_{\mathcal{P}} = (6, 9)$$

Pareto set: {A, B, C, D, F}, ideal: (1,3), nadir: (6,9)

Exercise 1.2 – Weighting Method (Simple)

Exercise 1.2

$$\max\ z_1 = 3x_1 + 2x_2, \qquad \max\ z_2 = x_1 + 4x_2$$

$$\begin{aligned} x_1 + x_2 &\leq 4 \\ 2x_1 + x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Tasks:

(a) Solve for the weights $\mathbf{w} = (1, 0)$, $(0.5, 0.5)$, and $(0, 1)$.

(b) Determine the ideal point.

(c) Which vertices of the feasible region are Pareto-optimal?

Solution: Exercise 1.2

(a) Weighting method:
Scalarized: $\max\ w(3x_1+2x_2)+(1-w)(x_1+4x_2) = \max\ (2w+1)x_1+(4-2w)x_2$

$w = 1$**:** $\max\ 3x_1 + 2x_2$. Vertices: $(0, 0)$, $(3, 0)$, $(2, 2)$, $(0, 4)$.
Objective values: 0, 9, 10, 8. **Optimum:** $(2, 2)$**,** $z_1 = 10$**,** $z_2 = 10$.
$w = 0.5$**:** $\max\ 2x_1 + 3x_2$.

Values: 0, 6, 10, 12. **Optimum:** $(0,4)$**,** $z_1 = 8$**,** $z_2 = 16$.
$w = 0$**:** $\max\ x_1 + 4x_2$.
Values: 0, 3, 10, 16. **Optimum:** $(0,4)$**,** $z_1 = 8$**,** $z_2 = 16$.
(b) Ideal point:
$z_1^{\max} = 10$ (at $(2,2)$), $z_2^{\max} = 16$ (at $(0,4)$).

$$\mathbf{z}^{\mathrm{id}} = (10,\ 16) \quad \text{(not simultaneously achievable)}$$

(c) Pareto-optimal vertices:
Vertex $(0,0)$: $z = (0,0)$ – dominated by all others.
Vertex $(3,0)$: $z = (9,3)$ – dominated by $(2,2)$ with $z = (10,10)$.
Vertex $(2,2)$: $z = (10,10)$ – Pareto-optimal (best z_1).
Vertex $(0,4)$: $z = (8,16)$ – Pareto-optimal (best z_2).
Pareto-optimal vertices: $(2,2)$ **and** $(0,4)$**. The edge between them is also Pareto-optimal.**

2 Level 2: Medium

Exercise 2.1 – Weighting Method vs. ε-Constraint

Exercise 2.1

$$\max\ z_1 = 5x_1 + 4x_2, \qquad \min\ z_2 = 3x_1 + x_2$$

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 \\ x_1 + 2x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

This is the example from the theory section.

(a) Calculate three Pareto points using the weighting method (choose $w \in \{0.2;\ 0.5;\ 0.8\}$).

(b) Calculate three Pareto points using the ε-constraint method ($\varepsilon \in \{0,\ 5,\ 10.5\}$, primary objective: max z_1).

(c) Compare the results of both methods.

(d) Calculate the ideal and nadir point.

Solution: Exercise 2.1

(a) Weighting method:
Transformation: max z_1, min $z_2 \Leftrightarrow$ max $(-z_2)$. Weighted: max $w \cdot z_1 + (1 - w) \cdot (-z_2)$.

$$\max\ w(5x_1 + 4x_2) - (1 - w)(3x_1 + x_2) = \max\ (8w - 3)x_1 + (5w - 1)x_2$$

$w = 0.2$: coefficients $(-1.4;\ 0)$. Optimum at $x_1 = 0$: test $(0, 3)$: $z_1 = 12$, $z_2 = 3$.
$w = 0.5$: coefficients $(1;\ 1.5)$. Maximizing at the vertices: $(0, 3)$: $0 + 4.5 = 4.5$; $(3, 1.5)$: $3 + 2.25 = 5.25$. Optimum: $(3, 1.5)$, $z_1 = 21$, $z_2 = 10.5$.
$w = 0.8$: coefficients $(3.4;\ 3)$. Optimum at $(3, 1.5)$: $z_1 = 21$, $z_2 = 10.5$.

w	(x_1^*, x_2^*)	z_1	z_2
0.2	$(0, 3)$	12	3
0.5	$(3, 1.5)$	21	10.5
0.8	$(3, 1.5)$	21	10.5

(b) ε-constraint: Primary: max z_1, secondary constraint: $z_2 = 3x_1 + x_2 \leq \varepsilon$.
$\varepsilon = 0$: $3x_1 + x_2 \leq 0 \Rightarrow x_1 = x_2 = 0$. $z_1 = 0$, $z_2 = 0$.
$\varepsilon = 5$: additional constraint $3x_1 + x_2 \leq 5$. The LP gives $x_1 = 1.5$, $x_2 = 0.5$: $z_1 = 9.5$, $z_2 = 5$.
$\varepsilon = 10.5$: equation $3x_1 + x_2 = 10.5$ together with $6x_1 + 4x_2 = 24$ and $x_1 + 2x_2 = 6$: $x_1 = 3$, $x_2 = 1.5$. $z_1 = 21$, $z_2 = 10.5$.

(c) Comparison: Both methods reach the extreme points $(0,0)$ with $z=(0,0)$, $(0,3)$ with $z=(12,3)$, and $(3,1.5)$ with $z=(21,10.5)$. The ε-constraint method additionally finds intermediate points on the Pareto front. The weighting method gives the same point for $w=0.5$ and $w=0.8$, since the front is piecewise linear in this region.

(d) Ideal and nadir point:

$$\mathbf{z}^{\text{id}} = (21,\ 0), \qquad \mathbf{z}^{\text{nad}} = (0,\ 10.5)$$

Pareto front: line segment from $(z_1, z_2) = (0,0)$ through $(12,3)$ to $(21,10.5)$.

Exercise 2.2 – Normalization and Compromise Solution

Exercise 2.2

The Pareto front of a bi-criteria LP (both objectives to be maximized) is given with the following Pareto-optimal points:

Point	z_1	z_2
P_1	20	5
P_2	15	10
P_3	10	16
P_4	4	18

(a) Determine the ideal and nadir point.

(b) Normalize the objective values to $[0,1]$.

(c) Calculate the Euclidean distance to the ideal point (in normalized space) for each Pareto point.

(d) Which point is the best compromise solution (minimum distance)?

(e) Alternatively calculate the Chebyshev metric (L_∞ distance) and compare.

Solution: Exercise 2.2

(a) Ideal and nadir point:

$$\mathbf{z}^{\text{id}} = (20, 18), \qquad \mathbf{z}^{\text{nad}} = (4, 5)$$

(b) Normalization: $\tilde{z}_i = \frac{z_i - z_i^{\text{nad}}}{z_i^{\text{id}} - z_i^{\text{nad}}}$

Point	$\tilde{z}_1 = \frac{z_1-4}{16}$	$\tilde{z}_2 = \frac{z_2-5}{13}$
P_1	$\frac{16}{16} = 1.000$	$\frac{0}{13} = 0.000$
P_2	$\frac{11}{16} = 0.688$	$\frac{5}{13} = 0.385$
P_3	$\frac{6}{16} = 0.375$	$\frac{11}{13} = 0.846$
P_4	$\frac{0}{16} = 0.000$	$\frac{13}{13} = 1.000$

(c) Euclidean distance to the ideal point $\tilde{\mathbf{z}}^{\text{id}} = (1, 1)$:

$$d_2(P_i) = \sqrt{(1-\tilde{z}_1)^2 + (1-\tilde{z}_2)^2}$$

$$d_2(P_1) = \sqrt{0^2 + 1^2} = 1.000$$
$$d_2(P_2) = \sqrt{0.312^2 + 0.615^2} = \sqrt{0.097 + 0.378} = \sqrt{0.475} \approx \mathbf{0.689}$$
$$d_2(P_3) = \sqrt{0.625^2 + 0.154^2} = \sqrt{0.391 + 0.024} = \sqrt{0.415} \approx 0.644$$
$$d_2(P_4) = \sqrt{1^2 + 0^2} = 1.000$$

(d) Best compromise solution: P_3 with $d_2 \approx 0.644$ (smallest distance to the ideal).

(e) Chebyshev metric: $d_\infty(P_i) = \max(1-\tilde{z}_1, 1-\tilde{z}_2)$

Point	$1-\tilde{z}_1$	$1-\tilde{z}_2$	d_∞
P_1	0.000	1.000	1.000
P_2	0.312	0.615	0.615
P_3	0.625	0.154	0.625
P_4	1.000	0.000	1.000

Chebyshev selects P_2 ($d_\infty = 0.615$), Euclidean selects P_3. The choice of metric affects the selected solution!

Euclidean: P_3; Chebyshev: P_2. Recommendation: P_2 or P_3 depending on preference.

3 Level 3: Hard

Exercise 3.1 – Three Objective Functions: Complete Analysis

Exercise 3.1

$$\max\ z_1 = 5x_1 + 4x_2, \quad \max\ z_2 = 2x_1 + 6x_2, \quad \min\ z_3 = 3x_1 + x_2$$

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 \\ x_1 + 2x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(a) Calculate the objective values at all four vertices of the feasible region: $(0,0)$, $(4,0)$, $(3,1.5)$, $(0,3)$.

(b) Check each pair of vertices for dominance.

(c) Identify the Pareto-optimal vertices.

(d) Calculate the ideal point.

(e) Which vertex is the best compromise solution (Chebyshev, equal weights)?

Solution: Exercise 3.1

(a) Objective values at the vertices:

Vertex	x_1	x_2	(z_1, z_2, z_3)
E_0	0	0	(0, 0, 0)
E_1	4	0	(20, 8, 12)
E_2	3	1.5	(21, 15, 10.5)
E_3	0	3	(12, 18, 3)

(b) Dominance check (minimization view: z_3 min, $-z_1$ min, $-z_2$ min):
E_0 vs. E_1: E_1 better in z_1 and z_2, worse in z_3. No dominance.
E_0 vs. E_2: E_2 better in z_1 and z_2, worse in z_3 (0 vs. 10.5). No dominance.
E_0 vs. E_3: E_3 better in z_1 and z_2, worse in z_3 (0 vs. 3). No dominance.
E_1 vs. E_2: z_1: $20 < 21$ (E1 worse), z_2: $8 < 15$ (E1 worse), z_3: $12 > 10.5$ (E1 worse). E_2 dominates E_1!
E_1 vs. E_3: z_1: $20 > 12$ (E1 better), z_2: $8 < 18$ (E1 worse). No dominance.
E_2 vs. E_3: z_1: $21 > 12$ (E2 better), z_2: $15 < 18$ (E2 worse). No dominance.
(c) Pareto-optimal vertices: E_1 is dominated by E_2 $\Rightarrow$ **Pareto vertices:** E_0, E_2, E_3.
(Note: E_0 with $z = (0,0,0)$ may reasonably be excluded depending on the application, but is formally Pareto-optimal.)
(d) Ideal point:

$$z_1^{\text{id}} = 21\ (E_2), \quad z_2^{\text{id}} = 18\ (E_3), \quad z_3^{\text{id}} = 0\ (E_0)$$

$$\mathbf{z}^{\mathrm{id}} = (21,\ 18,\ 0) \quad \text{(not simultaneously achievable)}$$

(e) Chebyshev compromise solution: Normalize with nadir $(0, 0, 12)$:

$$\tilde{z}_1 = z_1/21, \quad \tilde{z}_2 = z_2/18, \quad \tilde{z}_3 = (12 - z_3)/12$$

Vertex	$\tilde{z}_1$	$\tilde{z}_2$	$\tilde{z}_3$	$d_\infty = \max(1 - \tilde{z}_i)$
E_0	0	0	1	1.000
E_2	1	0.833	0.125	0.167
E_3	0.571	1	0.750	0.429

Best compromise solution: $E_2 = (3, 1.5)$ **with** $d_\infty = 0.167$.

Exercise 3.2 – Non-Convex Pareto Front and Method Comparison

Exercise 3.2

Consider a bi-criteria problem whose Pareto front contains the following (discrete) points:

Point	z_1 (max)	z_2 (max)
Q_1	0	10
Q_2	3	9
Q_3	7	7
Q_4	9	5
Q_5	10	0

There is additionally a point $Q^* = (5, 8)$ which is *not* found by the weighting method.

(a) Verify that Q^* is Pareto-optimal.

(b) Show that no weight vector $\mathbf{w} = (w, 1 - w)$ with $w \in [0, 1]$ produces the solution Q^* (non-convex front).

(c) Calculate which ε value you must choose to find Q^* using the ε-constraint method (primary: max z_1, secondary constraint: $z_2 \geq \varepsilon$).

(d) Calculate the ideal and nadir point of the entire Pareto front.

(e) Normalize all points and determine the best compromise solution (Euclidean distance).

Solution: Exercise 3.2

(a) Pareto optimality of $Q^* = (5, 8)$:
Check all other points: $Q_1 = (0, 10)$: $z_1(Q_1) = 0 < 5 \Rightarrow Q_1$ is worse in z_1. No dominance. $Q_2 = (3, 9)$: $z_1(Q_2) = 3 < 5 \Rightarrow$ worse in z_1. No dominance. $Q_3 = (7, 7)$: $z_1(Q_3) = 7 > 5$, $z_2(Q_3) = 7 < 8 \Rightarrow$ not comparable. $Q_4 = (9, 5)$: $z_1 > 5$, $z_2 < 8 \Rightarrow$ not comparable. $Q_5 = (10, 0)$: $z_2 < 8 \Rightarrow$ not comparable.
No point dominates $Q^* \Rightarrow Q^*$ **is Pareto-optimal.**
(b) Unreachability via the weighting method:
Weighted problem: $\max\ w \cdot z_1 + (1 - w) \cdot z_2$. For Q^* this gives the value $5w + 8(1 - w) = 8 - 3w$.
For Q^* to be optimal, we need: $8 - 3w \geq$ value of all other points:

$$8 - 3w \geq 10 - 10w \Rightarrow 7w \geq 2 \Rightarrow w \geq 2/7 \approx 0.286 \quad (Q_1)$$
$$8 - 3w \geq 7 + 0 \cdot w = 7 + 7(1 - w) + 7w \Rightarrow 8 - 3w \geq 7 + 7 - 7w \Rightarrow 4w \geq 6 \Rightarrow w \geq 1.5 \quad (Q_3, \text{imp}$$

For $Q_3 = (7, 7)$: $7w + 7(1 - w) = 7$ for all w. For Q^*: $5w + 8(1 - w) = 8 - 3w$. $8 - 3w > 7 \Rightarrow w < 1/3$. But for $Q_4 = (9, 5)$: $9w + 5(1 - w) = 5 + 4w > 8 - 3w \Leftrightarrow 7w > 3 \Leftrightarrow w > 3/7$.
Since $w < 1/3 \approx 0.333$ and $w > 3/7 \approx 0.429$ cannot hold simultaneously, **there is no w for which Q^* is optimal.** The Pareto front is non-convex in this region.
(c) ε-constraint: Set $z_2 \geq \varepsilon$ and max z_1.
Q^* has $z_2 = 8$. Choose $\varepsilon = 8$: then z_1 is maximized subject to $z_2 \geq 8$.
The points with $z_2 \geq 8$ are: $Q_1(0, 10)$, $Q_2(3, 9)$, $Q^*(5, 8)$.
Maximum z_1 among these: Q^* with $z_1 = 5$. $\varepsilon = 8$.
(d) Ideal and nadir point (including Q^*):

$$\mathbf{z}^{\text{id}} = (10,\ 10), \qquad \mathbf{z}^{\text{nad}} = (0,\ 0)$$

(e) Normalization and compromise solution:
Since nadir $= (0, 0)$ and ideal $= (10, 10)$: $\tilde{z}_i = z_i/10$.

Point	$\tilde{z}_1$	$\tilde{z}_2$	$1 - \tilde{z}_1$	$1 - \tilde{z}_2$	d_2
Q_1	0.0	1.0	1.0	0.0	1.000
Q_2	0.3	0.9	0.7	0.1	0.707
Q^*	0.5	0.8	0.5	0.2	**0.539**
Q_3	0.7	0.7	0.3	0.3	0.424
Q_4	0.9	0.5	0.1	0.5	0.510
Q_5	1.0	0.0	0.0	1.0	1.000

Best compromise solution: $Q_3 = (7, 7)$ with $d_2 = 0.424$ – the most balanced point.
Q^* ranks third with $d_2 = 0.539$, but would only be reachable via the ε-constraint method.