Game Theory

Module 3: Non-Zero-Sum Games

Nash Equilibrium, Pareto Efficiency and Social Dilemmas

English

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Complete introduction with derivation of payoff matrices, step-by-step analysis and three worked examples

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1 What Is a Non-Zero-Sum Game?

In a **non-zero-sum game**, the gains of both players are interdependent – but what one gains does *not* have to be lost by the other. Both players can win simultaneously, both can lose, or one can gain more than the other loses.

Definition: Non-Zero-Sum Game

A two-player game is a **non-zero-sum game** if for at least one strategy combination:

$$u_1(s_1, s_2) + u_2(s_1, s_2) \neq 0$$

The total payoff varies across strategy combinations. Both players can simultaneously gain or lose.

Comparison with Zero-Sum Games

	Zero-Sum Game	**Non-Zero-Sum Game**
Sum	always 0	varies
Interests	completely opposed	partly opposed, partly shared
Cooperation	impossible	possible and often beneficial
Examples	chess, poker	trade, negotiation, environment

The central questions in non-zero-sum games:

- Where is the **Nash equilibrium** – and is it stable?
- Is the Nash equilibrium **Pareto-efficient**, or do better outcomes exist for both?
- Is there a **social dilemma** – individual rationality leads to a collectively bad outcome?
- Is there a **coordination problem** – multiple equilibria, and how do players coordinate?

2 Key Concepts

Definition: Pareto Dominance and Pareto Efficiency

An outcome (s_1, s_2) **Pareto-dominates** (s_1', s_2') if:

$$u_1(s_1, s_2) \geq u_1(s_1', s_2') \quad \text{and} \quad u_2(s_1, s_2) \geq u_2(s_1', s_2')$$

with at least one strict inequality.
An outcome is **Pareto-efficient** (Pareto-optimal) if it is not Pareto-dominated by any other outcome: no player can be made better off without making the other worse off.

Definition: Social Dilemma

A **social dilemma** occurs when the Nash equilibrium is *not* Pareto-efficient: individual rationality leads to a collectively suboptimal result.

3 Example 1: The Prisoner's Dilemma

3.1 The Game Situation

Scenario: Two Suspects

Two suspects are interrogated separately. Each decides without knowing what the other will do:

- **Stay silent** (cooperate): no testimony against the other
- **Confess** (defect): testify against the other

3.2 Deriving the Payoff Matrix

Payoffs are years in prison (negative = prison time, 0 = released):

Combination	What happens?	Payoff (A, B)
Silent – Silent	No confession. Limited evidence. Both receive a light sentence.	$(-1, -1)$
Silent – Confess	B confesses, A stays silent. B goes free (crown witness), A receives the full sentence.	$(-10, 0)$
Confess – Silent	A confesses, B stays silent. A goes free, B receives the full sentence.	$(0, -10)$
Confess – Confess	Both confess. Mutual incrimination – both receive a medium sentence.	$(-6, -6)$

3.3 The Payoff Matrix

Suspect A \ Suspect B	Stay silent	Confess
Stay silent	$(-1, -1)$	$(-10, 0)$
Confess	$(0, -10)$	$(-6, -6)$ ★ Nash Eq.

3.4 Step 1: Best Responses

Best responses of Suspect A:

	B stays silent	B confesses	Best response of A
A stays silent	-1	-10	
A confesses	**0**	**-6**	**Confess** (always better!)

Regardless of what B does: $0 > -1$ (if B stays silent) and $-6 > -10$ (if B confesses).
$\Rightarrow$ **Confessing is strictly dominant for A.**

The same analysis applies symmetrically for B. $\Rightarrow$ **Confessing is strictly dominant for B.**

3.5 Step 2: Nash Equilibrium

Since confessing is dominant for both, the unique Nash equilibrium is:

$$\text{Nash Eq.:} \quad (\text{Confess}, \text{Confess}) = (-6, -6)$$

Check: Can A improve if B confesses? A stays silent: $-10 < -6$. No. ✓ Can B improve if A confesses? B stays silent: $-10 < -6$. No. ✓

3.6 Step 3: Pareto Analysis

Outcome	A	B	Sum	Pareto?
(Silent, Silent)	-1	-1	-2	**Pareto-efficient**
(Silent, Confess)	-10	0	-10	Pareto-efficient
(Confess, Silent)	0	-10	-10	Pareto-efficient
(Confess, Confess)	-6	-6	-12	**Pareto-dominated!**

The Nash equilibrium $(-6, -6)$ is Pareto-dominated by $(-1, -1)$:

$$-1 > -6 \quad \text{(A better off)} \quad \text{and} \quad -1 > -6 \quad \text{(B better off)}$$

The Dilemma: The Nash equilibrium is *not* Pareto-efficient. Both would be better off with (Silent, Silent) $= (-1, -1)$ – but each has an incentive to secretly confess and betray the other. Without trust, both end up in the worst outcome.

Possible solutions:

- *Repeated games:* In long-term relationships, cooperation and trust can emerge. "Tit-for-Tat": cooperate as long as the other cooperates.
- *Binding agreement:* A contract with a penalty for defection.
- *External authority:* Joint legal representation, union, regulator.

4 Example 2: Coordination Game – Technology Standard

4.1 The Game Situation

Scenario: Two Firms Choose an Industry Standard

Firms X and Y are launching new devices. Each must choose a technical standard: **Standard A** or **Standard B**. Products with different standards are *incompatible* – customers only buy if both firms support the same standard.

4.2 Deriving the Payoff Matrix

Combination	What happens?	Payoff (X, Y)
Standard A – Standard A	Full compatibility. Large market. Standard A is newer – higher payoff.	(4, 4)
Standard A – Standard B	Incompatible. Customers do not buy. Both get nothing.	(0, 0)
Standard B – Standard A	Incompatible. No market.	(0, 0)
Standard B – Standard B	Compatible. Standard B is older/cheaper – slightly smaller market.	(3, 3)

4.3 The Payoff Matrix

Firm X \ Firm Y	Standard A	Standard B
Standard A	(4, 4) ★ Nash Eq. 1	(0, 0)
Standard B	(0, 0)	(3, 3) ★ Nash Eq. 2

4.4 Step 1: Best Responses

Best responses of Firm X:

- If Y chooses Standard A: $4 > 0 \Rightarrow$ X chooses Standard A
- If Y chooses Standard B: $3 > 0 \Rightarrow$ X chooses Standard B

X has no dominant strategy – the best response depends on what Y does. The same applies symmetrically for Y.

4.5 Step 2: Nash Equilibria

There are **two** Nash equilibria:

- **(Standard A, Standard A)** $= (4, 4)$: Neither wants to deviate (deviation $\rightarrow 0 < 4$).
- **(Standard B, Standard B)** $= (3, 3)$: Neither wants to deviate (deviation $\rightarrow 0 < 3$).

4.6 Step 3: Pareto Analysis

Both Nash equilibria are Pareto-efficient – no social dilemma. But (A,A) $= (4, 4)$ is *better* for both than (B,B) $= (3, 3)$.

The Coordination Problem: Both equilibria are stable, but one is strictly better. Without communication, both may end up at $(0, 0)$ – the worst outcome. Even knowing that (A,A) is better: if X is uncertain whether Y will choose A, X might prefer B as the "safer" option if Y also picks B.

Possible solutions:

- *Communication:* Direct agreement – "We go with Standard A."
- *Focal point:* Which standard is "natural"? (Schelling point: the newer Standard A)
- *Industry norm:* A trade association designates Standard A.
- *First-mover:* Publicly announcing Standard A first forces the other to coordinate.

5 Example 3: Price Negotiation

5.1 The Game Situation

Scenario: Buyer and Seller

A buyer and a seller negotiate over the price of a product. Both can choose **High Price**, **Medium Price** or **No Purchase / No Sale**. If they agree, a trade occurs and the gains are split differently.

5.2 The Payoff Matrix

Buyer \ Seller	High Price	Medium Price	No Sale
High Price	(3, 5)	(2, 3)	(0, 0)
Medium Price	(4, 2) ★ Nash Eq. 1	(3, 3) ★ Nash Eq. 2	(0, 0)
No Purchase	(0, 0)	(0, 0)	(1, 1) ★ Nash Eq. 3

Selected cell derivations:

- **(High, High) = (3, 5):** Agreement at high price. Seller gains more, buyer pays a lot – small gain for buyer, large for seller.
- **(Medium, Medium) = (3, 3):** Fair middle price. Balanced split of trade surplus.
- **(No Purchase, No Sale) = (1, 1):** No trade. Small alternative gain for both (other options, no time wasted).

5.3 Pareto Analysis

Outcome	B	S	Pareto?	Note
(High, High) = (3, 5)	3	5	**Efficient**	Not dominated
(Medium, High) = (4, 2)	4	2	**Efficient**	Nash Eq. 1
(High, Medium) = (2, 3)	2	3	Dominated	by (4,2) and (3,3)
(Medium, Medium) = (3, 3)	3	3	**Efficient**	Nash Eq. 2
(0, 0) outcomes	0	0	Dominated	by all others
(No, No) = (1, 1)	1	1	**Efficient**	Nash Eq. 3

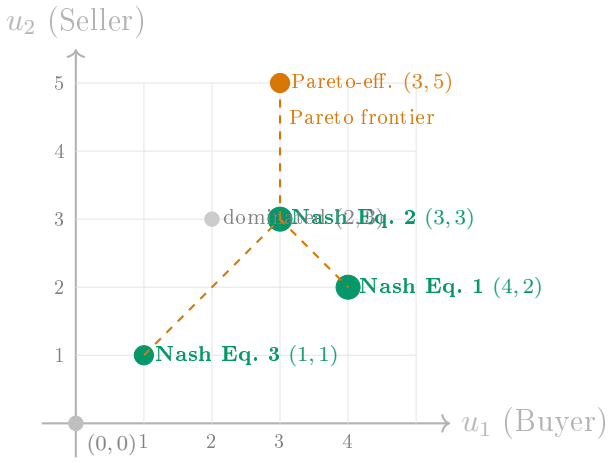
Interpretation:

- Nash Eq. 3 (No Purchase, No Sale) = (1, 1) is Pareto-efficient but clearly worse than Eq. 1 and 2 – trade would benefit both.
- The best outcome for the seller (3, 5) is not a Nash equilibrium: the buyer would switch to Medium Price and earn 4 instead of 3.
- The fairest outcome (3, 3) is a Nash equilibrium – a stable equilibrium under balanced bargaining power.

6 Theory: Payoff Space and Pareto Frontier

Definition: Payoff Space

The **payoff space** shows all possible outcome points (u_1, u_2) in a coordinate system. The axes measure the payoff of each player.



The **Pareto frontier** (dashed) connects all Pareto-efficient points. Points below the frontier are dominated – there always exists a better outcome for both players.

7 Tool Description: Game Theory Module 3

Non-Zero-Sum Game Tool

The tool computes Nash equilibria, Pareto efficiency and classifies the game.

7.1 Input

- Each cell: (**Payoff P1**, **Payoff P2**) – both payoffs.
- Rows = strategies of Player 1, Columns = strategies of Player 2.
- CSV format: `a,b` per cell (payoff P1, payoff P2).

7.2 Output – 5 Steps

1. **Best Responses:** For each strategy of the opponent, the optimal counter-strategy – with concrete payoffs.
2. **Nash Equilibria:** All pure Nash equilibria with verification. Warning when multiple equilibria exist (coordination problem).
3. **Pareto Efficiency:** Which outcomes are efficient? Which are dominated? Social dilemmas are automatically detected.
4. **Game Classification:** Zero-sum? Dilemma? Coordination?

5. **Interpretation:** Recommendations – Tit-for-Tat, contracts, focal points, first-mover strategy.

Payoff Space Graphic: All outcomes as points in the (u_1, u_2)-space. Green = Nash equilibrium, Orange = Pareto-efficient, Grey = dominated.

8 Summary

Key Points: Module 3 – Non-Zero-Sum Games

1. **Non-zero-sum:** Both can simultaneously win or lose – cooperation is possible and often beneficial.
2. A **Nash equilibrium** may be unique, multiple or absent (in pure strategies).
3. **Pareto efficiency:** An outcome is optimal if no player can be made better off without making the other worse off.
4. **Social dilemma:** Nash equilibrium is Pareto-inferior – individual rationality harms the collective. Solution: contracts, repetition, regulation.
5. **Coordination problem:** Multiple Nash equilibria – communication, focal points or norms required.
6. The **Prisoner's Dilemma** is the most famous social dilemma: confessing dominates, but staying silent is better for both.
7. The **payoff space** visualises all outcomes and the Pareto frontier compared to Nash equilibria.