Exercise Collection

Game Theory – Modules 1 to 4

English

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1 Module 1 – Fundamentals of Game Theory

Level 1: Easy

Exercise 1.1 – Reading the Matrix and Best Responses

Exercise 1.1

Two firms, Alpha and Beta, simultaneously decide on their advertising strategy: **Intensive** or **Minimal**. The payoff matrix shows profit in million euros (Alpha, Beta):

Alpha \ Beta	Intensive	Minimal
Intensive	(3, 3)	(5, 1)
Minimal	(1, 5)	(4, 4)

(a) Read the cell (Intensive, Minimal): what does each firm receive?

(b) Determine Alpha's best response to each of Beta's strategies.

(c) Determine Beta's best response to each of Alpha's strategies.

(d) Is there a Nash equilibrium in pure strategies?

Solution: Exercise 1.1

(a) Cell (Intensive, Minimal) = (5, 1): Alpha receives 5m €, Beta receives 1m €.
(b) Alpha's best responses:

- If Beta plays Intensive: $\max(3, 1) = 3 \Rightarrow$ Alpha chooses **Intensive**
- If Beta plays Minimal: $\max(5, 4) = 5 \Rightarrow$ Alpha chooses **Intensive**

Intensive is **strictly dominant** for Alpha.
(c) Beta's best responses:

- If Alpha plays Intensive: $\max(3, 1) = 3 \Rightarrow$ Beta chooses **Intensive**
- If Alpha plays Minimal: $\max(5, 4) = 5 \Rightarrow$ Beta chooses **Intensive**

Intensive is **strictly dominant** for Beta.
(d) Nash equilibrium: (Intensive, Intensive) = (3, 3). Both have Intensive as a dominant strategy. Check: Alpha cannot improve ($3 > 1$) ✓; Beta cannot improve ($3 > 1$) ✓.
Nash Eq.: (Intensive, Intensive) = (3, 3).
Note: (Minimal, Minimal) = (4, 4) would be better for both – a classic Prisoner's Dilemma.

Exercise 1.2 – Dominance and IEDS

Exercise 1.2

Consider the following 3×3 payoff matrix (Player A, Player B):

A \ B	Left	Centre	Right
Top	(4, 2)	(3, 3)	(1, 4)
Middle	(2, 4)	(2, 2)	(3, 1)
Bottom	(1, 1)	(1, 3)	(2, 2)

(a) Show that Bottom is dominated for A.

(b) Eliminate Bottom and then determine which strategy is dominated for B.

(c) Carry out IEDS completely.

(d) What is the Nash equilibrium?

Solution: Exercise 1.2

(a) Bottom is dominated by Top:

- Against Left: $4 > 1$ ✓; against Centre: $3 > 1$ ✓; against Right: $1 = 1$ – not strict.

Bottom is dominated by Middle: Left $2 > 1$, Centre $2 > 1$, Right $3 > 2$ ✓ – strictly dominated.

(b) After eliminating Bottom: remaining matrix:

	Left	Centre	Right
Top	(4, 2)	(3, 3)	(1, 4)
Middle	(2, 4)	(2, 2)	(3, 1)

For B: Left vs Right: $2 < 4$ (against Top) and $4 > 1$ (against Middle) – no dominance. Is Centre dominated by Left? Left: 2, 4; Centre: 3, 2 – no. Is Right dominated: against Top $4 > 1$, against Middle $4 > 1$ – Left weakly dominates Right; check strict: $2 > 4$? No. No strict dominance.
Closer check: Centre vs Right: Top $3 > 1$, Middle $2 > 1 \Rightarrow$ Centre strictly dominates Right!

(c) IEDS: Round 1: Bottom (dominated by Middle) → eliminate. Round 2: Right (dominated by Centre) → eliminate. Round 3: remaining 2×2:

	Left	Centre
Top	(4, 2)	(3, 3)
Middle	(2, 4)	(2, 2)

Round 3: Top dominates Middle: $4 > 2$, $3 > 2$ ✓. Round 4: Centre dominates Left: $3 > 2$ ✓.

(d) IEDS solution and Nash Eq.: **(Top, Centre) = (3, 3).**

Level 2: Medium

Exercise 1.3 – Finding Nash Equilibria

Exercise 1.3

Two mobile network operators (X, Y) decide on their network expansion effort: **High**, **Medium** or **Low**.

X \ Y	High	Medium	Low
High	(2, 2)	(4, 1)	(6, 0)
Medium	(1, 4)	(3, 3)	(5, 1)
Low	(0, 6)	(1, 5)	(4, 4)

(a) Determine all Nash equilibria.

(b) Is this a coordination problem, a Prisoner's Dilemma, or neither?

(c) Identify all Pareto-efficient outcomes.

(d) Which outcome would you recommend to cooperation-minded decision makers?

Solution: Exercise 1.3

(a) Best responses:
X: against High: $\max(2,1,0) = 2 \rightarrow$ High. Against Medium: $\max(4,3,1) = 4 \rightarrow$ High. Against Low: $\max(6,5,4) = 6 \rightarrow$ High. $\Rightarrow$ **High** is strictly dominant for X.
Y: symmetric (the matrix is symmetric): **High** is strictly dominant for Y.
Nash Eq.: (High, High) = (2, 2). Unique.
(b) The game is a **Prisoner's Dilemma**: The Nash Eq. (2,2) is Pareto-inferior. (Low, Low) = (4,4) would be better for both.
(c) Pareto-efficient outcomes (not dominated): $(6,0)$, $(4,1)$, $(3,3)$, $(1,4)$, $(0,6)$, $(5,1)$, $(1,5)$, $(4,4)$ – all outcomes on the Pareto frontier. Strictly Pareto-efficient: $(4,4)$ dominates $(2,2)$, and $(3,3)$ dominates $(2,2)$.
Pareto-dominated: $(2,2)$ is dominated by $(3,3)$, $(4,4)$, $(4,1)$, $(1,4)$.
(d) Recommendation: (Low, Low) = (4,4). Both benefit maximally from mutual cooperation. Requires a binding agreement or repeated play.

Exercise 1.4 – Coordination Game with Focal Point

Exercise 1.4

Two firms must agree, without communication, on a common IT format: **Format A**, **Format B** or **Format C**. Only matching choices yield a profit.

Firm 1 \ Firm 2	Format A	Format B	Format C
Format A	(5, 5)	(0, 0)	(0, 0)
Format B	(0, 0)	(3, 3)	(0, 0)
Format C	(0, 0)	(0, 0)	(2, 2)

(a) Determine all Nash equilibria.

(b) Are all Nash equilibria Pareto-efficient?

(c) What is a focal point (Schelling point), and which one would likely be chosen here?

(d) How could this coordination problem be resolved?

Solution: Exercise 1.4

(a) Best responses: if Y plays Format A, X's best response is Format A (5>0). Likewise for B and C. All three diagonal entries are Nash equilibria:

- **(A,A) = (5,5)**, **(B,B) = (3,3)**, **(C,C) = (2,2)**

(b) All three are Pareto-efficient (no other equilibrium improves both players). However, among the equilibria themselves: (A,A) Pareto-dominates (B,B) and (C,C).

(c) Focal point: A focal point is an equilibrium that, without communication, is recognised as "natural" – through symmetry, prominence, or convention. Here: **Format A** with payoff (5,5) is the focal point – it is the best equilibrium for both.

(d) Solution approaches:

- Direct communication: agree on Format A
- Industry standard: trade association sets Format A
- First-mover: one firm publicly announces Format A
- Market dominance: the larger firm sets the standard

Level 3: Hard

Exercise 1.5 – Complete Analysis with IEDS and Nash

Exercise 1.5

Two supermarket chains (A, B) compete with three pricing strategies each.

A \ B	High	Medium	Low
High	(6, 6)	(2, 8)	(1, 9)
Medium	(8, 2)	(4, 4)	(2, 7)
Low	(9, 1)	(7, 2)	(3, 3)

(a) Show in full that "Low" is strictly dominant for both players (proof with inequalities).

(b) Carry out IEDS and determine the Nash equilibrium.

(c) Compute the Pareto loss: how much do both players jointly lose compared to the Pareto-efficient alternative?

(d) Describe two concrete measures that could resolve the dilemma.

Solution: Exercise 1.5

(a) Dominance proof for A:

B's choice	Low	Medium	High	Is Low better?
B plays High	9	8	6	$9 > 8 > 6$ ✓
B plays Medium	7	4	2	$7 > 4 > 2$ ✓
B plays Low	3	2	1	$3 > 2 > 1$ ✓

Low strictly dominates Medium and High for A. By symmetry, the same holds for B.
(b) IEDS: Round 1: High and Medium are dominated for A $\rightarrow$ eliminate. Likewise for B. Solution: **(Low, Low) = (3, 3).**
(c) Pareto loss: Nash Eq.: $(3, 3)$, total welfare $= 6$. Pareto optimum (High, High) $= (6, 6)$, total welfare $= 12$. **Pareto loss:** $12 - 6 = 6$ **units** – both players lose 3 each compared to the cooperative solution.
(d) Resolution measures:

1. *Antitrust regulation (governmental):* Price-fixing agreements are illegal in practice, but a regulator could set minimum prices or restrict price transparency.

2. *Reputation and repetition:* In a repeated game, a "tit-for-tat" strategy could enforce cooperation: play High; if the opponent deviates, play Low as punishment.

Exercise 1.6 – Mixed Strategies in the Fundamentals Module

Exercise 1.6

An employee (E) decides daily: **Work** or **Shirk**. The employer (R) decides: **Monitor** or **Not monitor**.

E \ R	Monitor	Not monitor
Work	(3, 2)	(3, 4)
Shirk	(−2, 1)	(5, 0)

(a) Show that no pure Nash equilibrium exists.

(b) Compute the mixed strategy of R that makes E indifferent (probability q of monitoring).

(c) Compute the mixed strategy of E that makes R indifferent (probability p of working).

(d) Interpret the equilibrium: what does it mean practically that E works with probability p?

Solution: Exercise 1.6

(a) No pure Nash Eq.:

- (Work, Monitor): R could switch to Not monitor (4>2) → no equilibrium.
- (Work, Not monitor): E could switch to Shirk (5>3) → no equilibrium.
- (Shirk, Not monitor): R could switch to Monitor (1>0) → no equilibrium.
- (Shirk, Monitor): E could switch to Work (3>−2) → no equilibrium.

(b) Mixed strategy of R (q = prob. of Monitoring): E is made indifferent between Work and Shirk:

$$3q + 3(1-q) = -2q + 5(1-q) \;\Rightarrow\; 3 = 5 - 7q \;\Rightarrow\; \boldsymbol{q = \tfrac{2}{7} \approx 0.286}$$

(c) Mixed strategy of E (p = prob. of Working): R is made indifferent between Monitor and Not monitor:

$$2p + 1(1-p) = 4p + 0(1-p) \;\Rightarrow\; 1 + p = 4p \;\Rightarrow\; \boldsymbol{p = \tfrac{1}{3} \approx 0.333}$$

(d) Interpretation: In equilibrium, the employee works $\frac{1}{3}$ of the time and shirks $\frac{2}{3}$. The employer monitors $\frac{2}{7}$ of the time.

- Neither player can profit by changing their strategy.
- More monitoring would push the employee toward always working.
- Less monitoring would push the employee toward always shirking.

In practice: more frequent monitoring leads to more work effort – however, a deterministic monitoring schedule would be predictable and easy to circumvent.

2 Module 2 – Zero-Sum Games

Level 1: Easy

Exercise 2.1 – Identifying a Saddle Point

Exercise 2.1

Consider the following payoff matrix of a zero-sum game (payoff for Player 1, row maximizer):

P1 \ P2	Left	Centre	Right
Top	3	-1	4
Middle	2	5	1
Bottom	-2	3	2

(a) Compute the maximin (P1) and the minimax (P2).

(b) Does a saddle point exist?

(c) What is the game value?

(d) Which pure strategies do P1 and P2 play in equilibrium?

Solution: Exercise 2.1

(a) Row minima (Maximin P1):

- Top: $\min(3, -1, 4) = -1$
- Middle: $\min(2, 5, 1) = 1$
- Bottom: $\min(-2, 3, 2) = -2$

Maximin $= \max(-1, 1, -2) =$ **1** (strategy: Middle)
Column maxima (Minimax P2):

- Left: $\max(3, 2, -2) = 3$
- Centre: $\max(-1, 5, 3) = 5$
- Right: $\max(4, 1, 2) = 4$

Minimax $= \min(3, 5, 4) =$ **3** (strategy: Left)
(b) Maximin $= 1 \neq 3 =$ Minimax $\Rightarrow$ **No saddle point.**
(Cell (Middle, Left) $= 2$ is not a row minimum – 1 is. No saddle point confirmed.)
(c) The game value lies between 1 and 3; mixed strategies are required.
(d) No pure equilibrium; P1 chooses Middle as the maximin security strategy, P2 chooses Left as the minimax security strategy.

Exercise 2.2 – 2×2 Saddle Point

Exercise 2.2

An attacker (A) and a defender (D) in a war-game scenario. Payoff for A (attacker winning is positive):

A \ D	Defend North	Defend South
Attack North	-1	3
Attack South	4	-2

(a) Compute the maximin and minimax.

(b) Does a saddle point exist?

(c) Compute the optimal mixed strategy for A (p = prob. of attacking North).

(d) Compute the optimal mixed strategy for D (q = prob. of defending North) and the game value.

Solution: Exercise 2.2

(a) Maximin A: $\min(-1,3) = -1$; $\min(4,-2) = -2$. Maximin $= \max(-1,-2) = -1$. Minimax D: $\max(-1,4) = 4$; $\max(3,-2) = 3$. Minimax $= \min(4,3) = 3$.
(b) $-1 \neq 3 \Rightarrow$ **No saddle point.**
(c) Mixed strategy of A: D is made indifferent:

$$-1\cdot p + 4(1-p) = 3p + (-2)(1-p) \;\Rightarrow\; 4-5p = 5p-2 \;\Rightarrow\; 10p = 6 \;\Rightarrow\; \boldsymbol{p = 0.6}$$

(d) Mixed strategy of D: A is made indifferent:

$$-1\cdot q + 3(1-q) = 4q + (-2)(1-q) \;\Rightarrow\; 3-4q = 6q-2 \;\Rightarrow\; 10q = 5 \;\Rightarrow\; \boldsymbol{q = 0.5}$$

Game value: $v = -1\cdot 0.6 + 4\cdot 0.4 = -0.6 + 1.6 = \mathbf{1.0}$
Check: $3\cdot 0.6 + (-2)\cdot 0.4 = 1.8 - 0.8 = 1.0$ ✓

Level 2: Medium

Exercise 2.3 – Dominance in the Zero-Sum Game

Exercise 2.3

Consider the following 3×4 zero-sum game (payoff for row player):

	C1	C2	C3	C4
R1	4	2	6	3
R2	1	5	3	2
R3	3	3	4	4

(a) Eliminate dominated strategies of the column player (minimizer).

(b) Eliminate dominated strategies of the row player.

(c) Solve the remaining game (saddle point or mixed strategies).

(d) Compute the game value.

Solution: Exercise 2.3

(a) The column player minimizes. Compare columns (column player prefers lower values):
C1 vs C4: $4 > 3$, $1 < 2$, $3 < 4$ – no dominance. C2 vs C4: $2 < 3$, $5 > 2$, $3 < 4$ – no dominance. C3 vs C1: $6 > 4$, $3 > 1$, $4 > 3$ – C3 is worse for column than C1. Column player prefers low values: C1 $\leq$ C3 in every row? $4 < 6$, $1 < 3$, $3 < 4$ ✓ – **C3 is dominated by C1** (for column). Eliminate C3.
(b) Remaining (without C3):

	C1	C2	C4
R1	4	2	3
R2	1	5	2
R3	3	3	4

Row player maximizes: R2 vs R3: $1 < 3$, $5 > 3$, $2 < 4$ – no dominance. R1 vs R3: $4 > 3$, $2 < 3$, $3 < 4$ – no dominance.
(c) Maximin: R1: min=2; R2: min=1; R3: min=3. Maximin = 3 (R3). Minimax: C1: max=4; C2: max=5; C4: max=4. Minimax = 4 (C1 or C4).
$3 \neq 4$ – no saddle point. Mixed strategies required (solvable via the tool).
(d) The game value lies between 3 and 4.

Exercise 2.4 – Penalty Kick Scenario

Exercise 2.4

A striker takes a penalty against goalkeeper Keeper. Payoff = scoring probability for the striker (%):

Striker \ Keeper	Dive Left	Dive Right
Shoot left	30	80
Shoot right	75	25

(a) Show that no pure Nash equilibrium exists.

(b) Compute the striker's optimal mixed strategy (p = prob. of shooting left).

(c) Compute the keeper's optimal mixed strategy (q = prob. of diving left).

(d) What is the expected game value (expected scoring probability)?

(e) What happens if the striker always shoots left?

Solution: Exercise 2.4

(a) Checking pure equilibria: (Left, Left)=30: striker switches to Right (75>30) → no equilibrium. (Right, Left)=75: keeper switches to Right (25<75, minimizing) → no equilibrium. (Left, Right)=80: keeper switches to Left (30<80) → no equilibrium. (Right, Right)=25: striker switches to Left (80>25) → no equilibrium.
(b) Striker's mixed strategy (keeper indifferent):

$$30p+75(1-p) = 80p+25(1-p) \;\Rightarrow\; 75-45p = 25+55p \;\Rightarrow\; 100p = 50 \;\Rightarrow\; \mathbf{p = 0.5}$$

(c) Keeper's mixed strategy (striker indifferent):

$$30q+80(1-q) = 75q+25(1-q) \;\Rightarrow\; 80-50q = 25+50q \;\Rightarrow\; 100q = 55 \;\Rightarrow\; \boldsymbol{q = 0.55}$$

(d) $v = 30 \cdot 0.5 + 80 \cdot 0.5 = 55\%$ scoring probability.
(e) If the striker always shoots left: the keeper always dives left, scoring probability falls to just 30% – much worse than 55%. This shows: predictability is costly in zero-sum games.

Level 3: Hard

Exercise 2.5 – 3×3 Zero-Sum Game

Exercise 2.5

Two financial traders compete in a market. Payoff for Trader 1 (in thousand €):

T1 \ T2	Aggressive	Neutral	Defensive
Aggressive	0	6	8
Neutral	−6	0	4
Defensive	−8	−4	0

(a) Show that Defensive is dominated for T1.

(b) After elimination, show that Defensive is dominated for T2.

(c) Solve the remaining 2×2 game.

(d) Compute the game value and optimal mixed strategies.

(e) What does this game show about the advantage of aggressive behaviour?

Solution: Exercise 2.5

(a) Defensive vs Neutral for T1: $-8 < -6$, $-4 < 0$, $0 < 4$ – Defensive is strictly dominated by Neutral.
(b) After eliminating T1-Defensive:

	Aggressive	Neutral	Defensive
Aggressive	0	6	8
Neutral	-6	0	4

T2 minimizes. T2-Defensive vs T2-Neutral: $8 > 4$ and $4 > 0$ – T2 prefers Neutral (lower). T2-Aggressive vs T2-Neutral: $0 < 6$, $-6 < 0$ – T2 prefers Aggressive (lower). The game is antisymmetric: $a_{ij} = -a_{ji}$. T2-Defensive is worse than T2-Neutral for T2 (since $8 > 4 > 0$). T2-Defensive is strictly dominated by T2-Neutral: $8 > 4$, $4 > 0$ ✓.
(c) 2×2 reduced game:

	Aggressive	Neutral
Aggressive	0	6
Neutral	-6	0

Maximin T1: $\min(0, 6) = 0$; $\min(-6, 0) = -6$. Maximin $= 0$ (Aggressive). Minimax T2: $\max(0, -6) = 0$; $\max(6, 0) = 6$. Minimax $= 0$.
(d) Maximin = Minimax $= 0 \Rightarrow$ **Saddle point!** Cell (Aggressive, Aggressive) $= 0$. Game value $v^* = 0$. Both play Aggressive in equilibrium.
(e) The game shows: aggressive behaviour guarantees at least a draw (0). Defensive behaviour leads to negative outcomes. In zero-sum games without cooperation, aggression pays off.

Exercise 2.6 – Minimax Theorem and Sensitivity

Exercise 2.6

Consider the following 2×2 zero-sum game with parameter k:

$$A = \begin{pmatrix} k & 2 \\ 1 & k \end{pmatrix}$$

(a) For which values of k does a saddle point exist?

(b) For $k = 3$: compute the optimal mixed strategy and the game value.

(c) For $k = 0$: compute the optimal mixed strategy.

(d) Show that the game value lies between 1 and 2 for all $k \in (1, 2)$.

Solution: Exercise 2.6

(a) Saddle point: Maximin: $\min(k, 2)$ and $\min(1, k)$. Minimax: $\max(k, 1)$ and $\max(2, k)$.
A saddle point exists when Maximin = Minimax.

- $k \geq 2$: Maximin $= \min(k, 2), \min(1, k) = 2, 1$. Max $= 2$. Minimax $= \min(\max(k, 1), \max(2, k)) = \min(k, k) = k$. Equilibrium when $2 = k$, i.e. $k = 2$.
- $k \leq 1$: Maximin $= k$. Minimax $= \min(1, 2) = 1$. Equilibrium when $k = 1$.

Saddle point: $k = 1$ (cell 1,2 or 2,1) or $k = 2$ (cell 1,1 or 2,2).
(b) $k = 3$**:** Matrix $\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$. p (prob. of row 1): $3p + 1(1-p) = 2p + 3(1-p) \Rightarrow 1 + 2p = 3 - p \Rightarrow 3p = 2 \Rightarrow \boldsymbol{p = 2/3}$. Game value: $v = 3 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = 2 + \frac{1}{3} = \boldsymbol{\frac{7}{3}}$.
(c) $k = 0$**:** Matrix $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. $0 \cdot p + 1(1-p) = 2p + 0(1-p) \Rightarrow 1 - p = 2p \Rightarrow \boldsymbol{p = 1/3}$.
Game value: $v = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \boldsymbol{2/3}$.
(d) General game value for $k \in (1, 2)$: At $k = 1$: $v = 1$. At $k = 2$: $v = 2$. Since v is continuous in k between these boundary saddle points, $v \in (1, 2)$ holds for $k \in (1, 2)$.

3 Module 3 – Non-Zero-Sum Games

Level 1: Easy

Exercise 3.1 – Nash Equilibrium and Pareto in a Simple Game

Exercise 3.1

Two neighbours decide whether to maintain a shared garden (**Maintain**) or not (**Ignore**).

A \ B	Maintain	Ignore
Maintain	(4, 4)	(1, 5)
Ignore	(5, 1)	(2, 2)

(a) Determine all Nash equilibria.

(b) Is the outcome Pareto-efficient?

(c) What type of game is this?

(d) Which payoff would be best for both players?

Solution: Exercise 3.1

(a) Best responses: against Maintain: $\max(4,5) = 5 \rightarrow$ Ignore. Against Ignore: $\max(1,2) = 2 \rightarrow$ Ignore. Ignore is dominant for A (and symmetrically for B). **Nash Eq.: (Ignore, Ignore) = (2, 2).**
(b) (Ignore, Ignore) = (2,2) is Pareto-dominated by (Maintain, Maintain) = (4,4). $\Rightarrow$ **Not Pareto-efficient.**
(c) Prisoner's Dilemma: individually rational $\rightarrow$ collectively suboptimal.
(d) (Maintain, Maintain) = **(4, 4)** – both benefit. Requires coordination or a binding agreement.

Exercise 3.2 – Multiple Nash Equilibria

Exercise 3.2

Two drivers (A, B) at an intersection without traffic lights. Each decides: **Drive** or **Wait**.

A \ B	Drive	Wait
Drive	(−10, −10)	(3, 1)
Wait	(1, 3)	(0, 0)

(a) Determine all Nash equilibria.

(b) Are both equilibria Pareto-efficient?

(c) Why is this a coordination problem rather than a Prisoner's Dilemma?

(d) How could this problem be resolved in practice?

Solution: Exercise 3.2

(a)

- (Drive, Wait) = (3,1): A cannot improve ($3 > 1$), B cannot improve ($1 > -10$) ✓ – **Nash Eq. 1**.
- (Wait, Drive) = (1,3): A cannot improve ($1 > -10$), B cannot improve ($3 > 0$) ✓ – **Nash Eq. 2**.
- (Drive, Drive): A switches to Wait ($1 > -10$) → no equilibrium.
- (Wait, Wait): A switches to Drive ($3 > 0$) → no equilibrium.

(b) Both Nash equilibria (3,1) and (1,3) are Pareto-efficient (no outcome improves both).
(c) Not a Prisoner's Dilemma, because no player has a dominant strategy. It is a coordination problem: the players must coordinate, but have opposing preferences over the equilibria.
(d) Practical solutions: right-of-way rules, traffic lights, "first to arrive goes first" as a focal point.

Level 2: Medium

Exercise 3.3 – Bargaining Game

Exercise 3.3

A buyer (K) and a seller (V) negotiate over a house. K values it at 500, V has a reservation price of 300 (all values in thousand €). Both simultaneously choose an offer: **300**, **400** or **500**. Agreement occurs at matching offers; mismatched offers mean no trade. Payoff at agreement: K receives (500 − price), V receives (price − 300).

(a) Construct the complete payoff matrix.

(b) Identify all Nash equilibria.

(c) Are all Nash equilibria Pareto-efficient?

(d) Which Nash equilibrium would you call fairest, and why?

Solution: Exercise 3.3

(a) On agreement: price 300 → (200, 0); price 400 → (100, 100); price 500 → (0, 200). On disagreement: (0, 0).

K \ V	300	400	500
300	(200, 0)	(0, 0)	(0, 0)
400	(0, 0)	(100, 100)	(0, 0)
500	(0, 0)	(0, 0)	(0, 200)

(b) All three diagonal entries are Nash equilibria: **(300,300)=(200,0)**, **(400,400)=(100,100)**, **(500,500)=(0,200)**.
(c) All three are Pareto-efficient (disagreement (0,0) would be worse for both). But among themselves: (100,100) is fairer; (200,0) and (0,200) are extreme distributions.
(d) **(400, 400) = (100, 100)** is the fairest Nash equilibrium:

- Symmetric split of the trading surplus (200 thousand total, 100 each)
- Coincides with the Nash bargaining solution (product maximised: $100 \times 100 = 10000$)

Exercise 3.4 – Pareto Analysis and Social Optimum

Exercise 3.4

Two firms (X, Y) decide on environmental investment: **High**, **Medium** or **Low**.

X \ Y	High	Medium	Low
High	(4, 4)	(2, 5)	(0, 6)
Medium	(5, 2)	(3, 3)	(1, 4)
Low	(6, 0)	(4, 1)	(2, 2)

(a) Determine all Nash equilibria.

(b) Construct a complete Pareto table.

(c) Compute social welfare (sum of payoffs) for all outcomes.

(d) Which outcome maximises total welfare, and how does it differ from the Nash equilibrium?

Solution: Exercise 3.4

(a) Low is dominant for both. Nash Eq.: **(Low, Low) = (2, 2).**
(b/c) Pareto table with welfare:

Outcome	X	Y	Welfare	Pareto
(High, High)	4	4	8	efficient
(High, Medium)	2	5	7	dominated
(High, Low)	0	6	6	efficient
(Medium, High)	5	2	7	dominated
(Medium, Medium)	3	3	6	dominated
(Medium, Low)	1	4	5	dominated
(Low, High)	6	0	6	efficient
(Low, Medium)	4	1	5	dominated
(Low, Low)	2	2	4	dominated – **Nash Eq.!**

(d) Welfare maximum: **(High, High) = (4, 4), welfare = 8.** The Nash Eq. (2,2) has welfare 4 – only half! ⇒ Regulation (environmental tax, subsidies) could steer the firms towards the social optimum.

Level 3: Hard

Exercise 3.5 – Repeated Game and Tit-for-Tat

Exercise 3.5

Two oligopolists (A, B) play the following pricing game:

A \ B	High	Low
High	(8, 8)	(0, 12)
Low	(12, 0)	(3, 3)

The game is played **infinitely often**. Both players use discount factor $\delta \in (0, 1)$.

(a) Determine the Nash Eq. of the one-shot game.

(b) A uses Tit-for-Tat: start with High; in each subsequent period mimic the opponent's previous-period choice. Compute the present value of mutual cooperation (High/High) for A.

(c) Compute the present value of unilateral deviation (Low in $t = 0$, then Nash punishment from $t = 1$) for A.

(d) Show that cooperation is sustainable if $\delta \geq \frac{4}{9}$.

Solution: Exercise 3.5

(a) One-shot Nash Eq.: Low is dominant (12>8 and 3>0). **(Low, Low) = (3, 3).**

(b) Present value of cooperation (High/High forever):

$$V_K = 8 + 8\delta + 8\delta^2 + \cdots = \frac{8}{1-\delta}$$

(c) Present value of deviation (Low in $t = 0$, then Nash from $t = 1$):

$$V_A = 12 + 3\delta + 3\delta^2 + \cdots = 12 + \frac{3\delta}{1-\delta}$$

(d) Cooperation is stable when $V_K \geq V_A$:

$$\frac{8}{1-\delta} \geq 12 + \frac{3\delta}{1-\delta}$$

$$\frac{8-3\delta}{1-\delta} \geq 12 \;\Rightarrow\; 8-3\delta \geq 12(1-\delta) \;\Rightarrow\; 8-3\delta \geq 12-12\delta \;\Rightarrow\; 9\delta \geq 4 \;\Rightarrow\; \boldsymbol{\delta \geq \frac{4}{9}}$$

At $\delta \geq 4/9$, Tit-for-Tat is a Nash equilibrium strategy of the repeated game. In practice: both firms must sufficiently value the future.

Exercise 3.6 – Mixed Strategies in the Non-Zero-Sum Game

Exercise 3.6

Two players (R = Row, C = Column) play the following game:

R \ C	Left	Right
Top	(2, 1)	(0, 0)
Bottom	(0, 0)	(1, 2)

(a) Determine all pure Nash equilibria.

(b) Show that a mixed Nash equilibrium exists, and compute p^* (R plays Top with probability p) and q^* (C plays Left with probability q).

(c) Compute the payoffs in the mixed equilibrium for both players.

(d) Why is this game called a "Battle of the Sexes"? Compare the three Nash equilibria.

Solution: Exercise 3.6

(a) Pure Nash equilibria:

- (Top, Left) = (2,1): R cannot improve ($2 > 0$), C cannot improve ($1 > 0$) ✓
- (Bottom, Right) = (1,2): R cannot improve ($1 > 0$), C cannot improve ($2 > 0$) ✓

(b) Mixed equilibrium: C indifferent (q = prob. of Left): $2q + 0(1-q) = 0 \cdot q + 1(1-q) \Rightarrow 2q = 1 - q \Rightarrow \boldsymbol{q^* = 1/3}$. R indifferent ($p$ = prob. of Top): $1 \cdot p + 0(1-p) = 0 \cdot p + 2(1-p) \Rightarrow p = 2 - 2p \Rightarrow \boldsymbol{p^* = 2/3}$.

(c) Payoffs in the mixed equilibrium:

- R: $2 \cdot \frac{2}{3} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9} + \frac{2}{9} = \boldsymbol{\frac{2}{3}}$
- C: $1 \cdot \frac{2}{3} \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9} + \frac{4}{9} = \boldsymbol{\frac{2}{3}}$

(d) Battle of the Sexes: A coordination game with opposing preferences: R prefers (Top, Left) = (2,1), C prefers (Bottom, Right) = (1,2). Both wish to coordinate, but on different equilibria. Comparison of the three equilibria:

- (Top, Left): good for R (2), bad for C (1)
- (Bottom, Right): bad for R (1), good for C (2)
- Mixed: both receive only 2/3 – the worst outcome for both!

4 Module 4 – Cooperative Game Theory

Level 1: Easy

Exercise 4.1 – Shapley Value: 3 Players

Exercise 4.1

Three firms (1, 2, 3) can jointly carry out a project. The characteristic function describes achievable profit in thousand €:

$$v(\emptyset) = 0, \quad v(\{1\}) = 0, \quad v(\{2\}) = 0, \quad v(\{3\}) = 0$$

$$v(\{1,2\}) = 60, \quad v(\{1,3\}) = 40, \quad v(\{2,3\}) = 30, \quad v(\{1,2,3\}) = 90$$

(a) Compute the Shapley value for all three players.

(b) Check: does the Shapley allocation satisfy efficiency and individual rationality?

(c) Which player has the largest contribution to the grand coalition?

Solution: Exercise 4.1

(a) Shapley value: $\phi_i = \sum_{S \ni i} \frac{(|S|-1)!(n-|S|)!}{n!}[v(S) - v(S \setminus \{i\})]$
For $n = 3$: weight for $|S| = 1$: $\frac{0!\cdot 2!}{6} = \frac{2}{6}$; for $|S| = 2$: $\frac{1!\cdot 1!}{6} = \frac{1}{6}$; for $|S| = 3$: $\frac{2!\cdot 0!}{6} = \frac{2}{6}$.
Player 1:

$$\begin{aligned}\phi_1 &= \tfrac{2}{6}[v(\{1\}) - v(\emptyset)] + \tfrac{1}{6}[v(\{1,2\}) - v(\{2\})] + \tfrac{1}{6}[v(\{1,3\}) - v(\{3\})] + \tfrac{2}{6}[v(\{1,2,3\}) - v(\{2,3\})] \\ &= \tfrac{2}{6}\cdot 0 + \tfrac{1}{6}\cdot 60 + \tfrac{1}{6}\cdot 40 + \tfrac{2}{6}\cdot 60 \\ &= 0 + 10 + 6.67 + 20 = \mathbf{36.67}\end{aligned}$$

Player 2: $\phi_2 = \frac{2}{6}\cdot 0 + \frac{1}{6}\cdot 60 + \frac{1}{6}\cdot 30 + \frac{2}{6}\cdot 50 = 0 + 10 + 5 + 16.67 = \mathbf{31.67}$
Player 3: $\phi_3 = 90 - 36.67 - 31.67 = \mathbf{21.67}$
Check: $36.67 + 31.67 + 21.67 = 90$ ✓
(b) Efficiency: $\sum \phi_i = 90 = v(\{1,2,3\})$ ✓ Individual rationality: $36.67 > 0$, $31.67 > 0$, $21.67 > 0$ – all receive more than alone ✓.
(c) Player 1 has the largest contribution ($\phi_1 = 36.67$) – they create the most added value across coalitions.

Exercise 4.2 – Core and Stability

Exercise 4.2

Three players (1, 2, 3) with characteristic function:

$$v(\{1\}) = 10, \; v(\{2\}) = 20, \; v(\{3\}) = 15$$

$$v(\{1,2\}) = 50, \; v(\{1,3\}) = 40, \; v(\{2,3\}) = 45, \; v(\{1,2,3\}) = 70$$

(a) What is the Core of a cooperative game?

(b) Formulate all conditions a distribution (x_1, x_2, x_3) must satisfy to lie in the Core.

(c) Check whether $(20, 25, 25)$ lies in the Core.

(d) Compute the Shapley value and check whether it lies in the Core.

Solution: Exercise 4.2

(a) Core: The set of all distributions that no subgroup of players can improve upon by splitting off.
(b) Core conditions:

- Efficiency: $x_1 + x_2 + x_3 = 70$
- Individual rationality: $x_1 \geq 10$, $x_2 \geq 20$, $x_3 \geq 15$
- Coalition rationality: $x_1 + x_2 \geq 50$, $x_1 + x_3 \geq 40$, $x_2 + x_3 \geq 45$

(c) Checking $(20, 25, 25)$: Efficiency: $70 = 70$ ✓. Individual: $20 > 10$, $25 > 20$, $25 > 15$ ✓. Coalitions: $20 + 25 = 45 < 50$ **VIOLATED!** $\Rightarrow$ $(20, 25, 25)$ **does not lie in the Core**.
(d) Shapley value: $\phi_1 = \frac{2}{6}(10) + \frac{1}{6}(30) + \frac{1}{6}(25) + \frac{2}{6}(25) = 3.33 + 5 + 4.17 + 8.33 = 20.83$ $\phi_2 = \frac{2}{6}(20) + \frac{1}{6}(30) + \frac{1}{6}(30) + \frac{2}{6}(25) = 6.67 + 5 + 5 + 8.33 = 25$ $\phi_3 = 70 - 20.83 - 25 = 24.17$
Core check: $\phi_1 + \phi_2 = 45.83 < 50$ **VIOLATED** – the Shapley value **does not lie in the Core** (the Core can be empty).

Level 2: Medium

Exercise 4.3 – Shapley Value: Voting Weights

Exercise 4.3

A board has 4 members with voting weights: $w_1 = 3$, $w_2 = 2$, $w_3 = 2$, $w_4 = 1$. Decisions require a majority of $q = 5$ votes. The characteristic function is:

$$v(S) = 1 \text{ if } \sum_{i \in S} w_i \geq 5, \quad v(S) = 0 \text{ otherwise.}$$

(a) Compute $v(S)$ for all coalitions.

(b) Compute the Shapley value for all four players.

(c) Interpret: does player 4 (with 1 vote) really have no influence?

(d) Compare voting weights with the Shapley power index.

Solution: Exercise 4.3

(a) Winning coalitions (sum ≥ 5):

- Pairs: $\{1,2\} = 5\checkmark$, $\{1,3\} = 5\checkmark$, $\{2,3\} = 4\times$, $\{1,4\} = 4\times$, $\{2,4\} = 3\times$, $\{3,4\} = 3\times$
- Triples: $\{1,2,3\} = 7\checkmark$, $\{1,2,4\} = 6\checkmark$, $\{1,3,4\} = 6\checkmark$, $\{2,3,4\} = 5\checkmark$
- $\{1,2,3,4\} = 8\checkmark$

(b) Shapley values (over all $4! = 24$ permutations):
Each player receives $1/24$ for every ordering in which they are pivotal (turn the coalition from losing to winning).
Standard result: $\phi_1 = \frac{10}{24} \approx 0.417$; $\phi_2 = \phi_3 = \frac{6}{24} = 0.250$; $\phi_4 = \frac{2}{24} \approx 0.083$.
Check: $0.417 + 0.250 + 0.250 + 0.083 = 1$ ✓
(c) Player 4 has Shapley value 8.3% – they really do have influence, namely in situations where $\{2,3,4\}$ becomes winning. Without player 4, $\{2,3\} = 4 < 5$ – they are pivotal in this coalition.
(d) Voting weights: $3:2:2:1 = 37.5\% : 25\% : 25\% : 12.5\%$. Shapley: $41.7\% : 25\% : 25\% : 8.3\%$. Player 1 has more power than their vote share, player 4 has less.

Exercise 4.4 – Nash Bargaining Solution

Exercise 4.4

Two parties (A, B) negotiate over allocating a profit of 100 thousand €. The disagreement point (status quo) is $(d_A, d_B) = (10, 20)$.

(a) What is the Nash bargaining solution? Formulate the optimisation problem.

(b) Compute the Nash solution (x_A^*, x_B^*).

(c) How does the solution change if the status quo rises to $(20, 20)$?

(d) Interpret the axioms of the Nash bargaining solution (efficiency, symmetry, scale invariance, independence of irrelevant alternatives).

Solution: Exercise 4.4

(a) The Nash bargaining solution maximises the Nash product:

$$\max_{x_A, x_B} (x_A - d_A)(x_B - d_B) \quad \text{s.t.} \quad x_A + x_B = 100,\ x_A \geq d_A,\ x_B \geq d_B$$

(b) With $d_A = 10$, $d_B = 20$ and $x_A + x_B = 100$:

$$\max_{x_A} (x_A - 10)(80 - x_A)$$

FOC: $(80 - x_A) - (x_A - 10) = 0 \Rightarrow 90 = 2x_A \Rightarrow \boldsymbol{x_A^* = 45}$, $\boldsymbol{x_B^* = 55}$.

Interpretation: B has the higher status quo ($20 > 10$) and receives more ($55 > 45$).
(c) Status quo $(20, 20)$: $\max(x_A - 20)(x_B - 20)$, $x_A + x_B = 100$. FOC: $(80 - x_A) - (x_A - 20) = 0 \Rightarrow 100 - 2x_A = 0 \Rightarrow \boldsymbol{x_A^* = 50}$, $\boldsymbol{x_B^* = 50}$. Symmetric status quo $\Rightarrow$ symmetric split.
(d) Axioms:

- *Efficiency:* The solution lies on the Pareto frontier.
- *Symmetry:* In a symmetric game, both receive equal amounts.
- *Scale invariance:* Linear transformations do not change the solution.
- *IIA:* Irrelevant alternatives do not change the solution.

Level 3: Hard

Exercise 4.5 – Convex Game and Core

Exercise 4.5

A game (N, v) with $N = \{1, 2, 3\}$ and:

$$v(\{1\}) = 2,\ v(\{2\}) = 3,\ v(\{3\}) = 1$$

$$v(\{1,2\}) = 10,\ v(\{1,3\}) = 7,\ v(\{2,3\}) = 8,\ v(\{1,2,3\}) = 15$$

(a) Show that the game is convex (check the convexity condition for all relevant coalition pairs).

(b) Compute the Shapley value.

(c) Show that for convex games the Shapley value lies in the Core.

(d) Determine all vertices of the Core.

Solution: Exercise 4.5

(a) Convexity condition: $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all S, T.
Check $S = \{1,2\}, T = \{1,3\}$: $v(\{1,2,3\}) + v(\{1\}) \geq v(\{1,2\}) + v(\{1,3\})$: $15 + 2 \geq 10 + 7 \Rightarrow 17 \geq 17$ ✓
$S = \{1,2\}, T = \{2,3\}$: $v(\{1,2,3\}) + v(\{2\}) \geq v(\{1,2\}) + v(\{2,3\})$: $15 + 3 \geq 10 + 8 \Rightarrow 18 \geq 18$ ✓
$S = \{1,3\}, T = \{2,3\}$: $15 + v(\{3\}) \geq 7 + 8$: $15 + 1 = 16 \geq 15$ ✓
All further conditions are trivial (singletons). **The game is convex.**
(b) Shapley value: $\phi_1 = \frac{2}{6}(2) + \frac{1}{6}(7) + \frac{1}{6}(6) + \frac{2}{6}(7) = \frac{4}{6} + \frac{7}{6} + \frac{6}{6} + \frac{14}{6} = \frac{31}{6} \approx \mathbf{5.17}$
$\phi_2 = \frac{2}{6}(3) + \frac{1}{6}(8) + \frac{1}{6}(7) + \frac{2}{6}(7) = \frac{6}{6} + \frac{8}{6} + \frac{7}{6} + \frac{14}{6} = \frac{35}{6} \approx \mathbf{5.83}$ $\phi_3 = 15 - 5.17 - 5.83 = \mathbf{4.00}$
Check: $5.17 + 5.83 + 4 = 15$ ✓
(c) Property of convex games: for convex games the Shapley value always lies in

the Core, because supermodularity guarantees that the marginal contributions satisfy all Core conditions. Check: $\phi_1 + \phi_2 = 11 \geq 10$ ✓; $\phi_1 + \phi_3 = 9.17 \geq 7$ ✓; $\phi_2 + \phi_3 = 9.83 \geq 8$ ✓.
(d) Core vertices are obtained from the Shapley values across different orderings (for convex games these are exactly the Core's vertices): $(2, 8, 5)$, $(2, 7, 6)$, $(7, 3, 5)$, $(6, 8, 1)$, $(7, 7, 1)$, $(9, 5, 1)$ – one per permutation. The Shapley value $(5.17, 5.83, 4)$ is the centroid (average) of all vertices.

Exercise 4.6 – Fair Cost Allocation

Exercise 4.6

Three cities (A, B, C) want to jointly build a water pipeline. Costs for individual or combined pipelines (in million €):

$$c(\{A\}) = 12,\ c(\{B\}) = 10,\ c(\{C\}) = 8$$

$$c(\{A, B\}) = 16,\ c(\{A, C\}) = 15,\ c(\{B, C\}) = 13,\ c(\{A, B, C\}) = 20$$

(a) Show that the grand coalition is efficient (subadditive).

(b) Define the savings characteristic function $v(S) = \sum_{i \in S} c(\{i\}) - c(S)$ and compute all values.

(c) Compute the Shapley value of the savings.

(d) Compute the Shapley cost contribution of each city and check rationality.

Solution: Exercise 4.6

(a) Subadditivity: $c(\{A, B, C\}) = 20 \leq 12 + 10 + 8 = 30$ ✓ (savings of 10m €). Also: $c(\{A, B\}) = 16 \leq 12 + 10 = 22$ ✓; $c(\{A, C\}) = 15 \leq 20$ ✓; $c(\{B, C\}) = 13 \leq 18$ ✓.
(b) Savings function:

$$\begin{aligned} v(\{A\}) = 12 - 12 = 0, \quad v(\{B\}) = 0, \quad v(\{C\}) = 0 \\ v(\{A, B\}) = 12 + 10 - 16 = 6 \\ v(\{A, C\}) = 12 + 8 - 15 = 5 \\ v(\{B, C\}) = 10 + 8 - 13 = 5 \\ v(\{A, B, C\}) = 12 + 10 + 8 - 20 = 10 \end{aligned}$$

(c) Shapley value of the savings: $\phi_A = \frac{2}{6}(0) + \frac{1}{6}(6) + \frac{1}{6}(5) + \frac{2}{6}(5) = 0 + 1 + \frac{5}{6} + \frac{10}{6} = \frac{21}{6} = 3.5$ $\phi_B = \frac{2}{6}(0) + \frac{1}{6}(6) + \frac{1}{6}(5) + \frac{2}{6}(5) = 0 + 1 + \frac{5}{6} + \frac{10}{6} = 3.5$ $\phi_C = 10 - 3.5 - 3.5 = 3.0$
(d) Shapley cost contribution: City A pays: $c(\{A\}) - \phi_A = 12 - 3.5 =$ **8.5** million. City B pays: $c(\{B\}) - \phi_B = 10 - 3.5 =$ **6.5** million. City C pays: $c(\{C\}) - \phi_C = 8 - 3.0 =$ **5.0** million.
Check: $8.5 + 6.5 + 5.0 = 20$ million $= c(\{A, B, C\})$ ✓

Rationality: every city pays less than its standalone cost ($8.5 < 12$, $6.5 < 10$, $5.0 < 8$) ✓.
