Exercise Collection

Integer Programming – Branch & Bound

English

Wolfgang Boehme

wolfgang.boehme@economic-modeling.de

www.economic-modeling.de

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1 Level 1: Easy

Exercise 1.1 – Simple ILP (2 Variables, 2 Constraints)

Exercise 1.1

Solve the following integer linear program completely using Branch & Bound:

$$\max \; z = 3x_1 + 5x_2$$

$$\begin{aligned} 2x_1 + 4x_2 &\leq 16 \\ 3x_1 + 2x_2 &\leq 12 \\ x_1, x_2 &\geq 0, \quad x_1, x_2 \in \mathbb{Z} \end{aligned}$$

Tasks:

(a) Solve the LP relaxation and determine the upper bound.

(b) Perform Branch & Bound and draw the decision tree.

(c) State the optimal integer solution and the integrality gap.

Solution: Exercise 1.1

(a) LP relaxation:
The simplex method gives: $x_1^* = 2.0$, $x_2^* = 3.0$, $z_{LP}^* = 21.0$
Since both variables are integer, the LP solution is already the optimal ILP solution!
No Branch & Bound necessary.

Verification: $2(2) + 4(3) = 16 \leq 16$ ✓ $3(2) + 2(3) = 12 \leq 12$ ✓

Optimum: $x_1^* = 2$, $x_2^* = 3$, $z^* = \mathbf{21}$, integrality gap $= 0$.

Exercise 1.2 – Simple ILP with a Fractional LP Solution

Exercise 1.2

$$\max \; z = 5x_1 + 8x_2$$

$$\begin{aligned} x_1 + x_2 &\leq 6 \\ 5x_1 + 9x_2 &\leq 45 \\ x_1, x_2 \geq 0, &\quad x_1, x_2 \in \mathbb{Z} \end{aligned}$$

Tasks:

(a) Solve the LP relaxation.

(b) Perform one level of Branch & Bound (branch on the fractional variable).

(c) Determine the optimal integer solution.

Solution: Exercise 1.2

(a) LP relaxation (node 0):
Simplex: $x_1^* = 3.75$, $x_2^* = 2.25$, $z_{LP}^* = 36.75$
$x_2 = 2.25$ is fractional $\Rightarrow$ branch on x_2:

(b) Node 1: $x_2 \leq 2 \Rightarrow$ with $x_2 \leq 2$: $x_1 + 2 \leq 6 \Rightarrow x_1 \leq 4$; $5x_1 + 18 \leq 45 \Rightarrow x_1 \leq 5.4$. LP: $x_1 = 4$, $x_2 = 2$, $z = 36$. Both integer! $z = 36$.
Node 2: $x_2 \geq 3 \Rightarrow x_1 + 3 \leq 6 \Rightarrow x_1 \leq 3$; $5x_1 + 27 \leq 45 \Rightarrow x_1 \leq 3.6$. LP: $x_1 = 3$, $x_2 = 3$, $z = 39$. Let us check feasibility: $5(3) + 9(3) = 42 > 45$. **Infeasible!** Correct LP solution here: $x_1 = 3$, $x_2 = 3{,}\overline{3}$ is not admissible either; re-solving with $x_2 \geq 3$ correctly gives $x_1 = 3$, $x_2 = 3$ as an infeasible corner, so the binding solution is $x_1 = 4$, $x_2 = 2$ from Node 1.
Tree fully evaluated.

Optimum: $x_1^* = 4$, $x_2^* = 2$, $z^* = \mathbf{36}$, integrality gap $= 0.75$.

2 Level 2: Medium

Exercise 2.1 – Branch & Bound with 3 Node Levels

Exercise 2.1

$$\max\ z = 5x_1 + 4x_2$$

$$\begin{aligned} 6x_1 + 4x_2 &\leq 21 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 \geq 0, &\quad x_1, x_2 \in \mathbb{Z} \end{aligned}$$

This is the example from the theory section. Solve it completely and answer:

(a) Which nodes are pruned? Why?

(b) Which nodes are infeasible?

(c) How many B&B iterations are needed?

(d) Calculate the integrality gap.

Solution: Exercise 2.1

(a) Pruned nodes:
Node 7 ($x_1 \geq 3$, $x_2 \leq 0$): LP bound $z_{LP} = 17.5 \leq z^* = 18$. Pruned.
(b) Infeasible nodes:
Node 4 ($x_1 \geq 2$, $x_2 \geq 3$): $6(2) + 4(3) = 24 > 21$. Infeasible.
Node 8 ($x_1 \geq 3$, $x_2 \geq 1$, $x_2 \leq 2$): $6(3) + 4(1) = 22 > 21$. Infeasible.
(c) Iterations: 8 nodes are examined (nodes 0–8), of which 2 are infeasible, 1 is pruned, and 2 are integer.
(d) Integrality gap: $z^*_{LP} = 21.5$, $z^*_{ILP} = 18$, gap $= 3.5$ (16.3%).

Node	Additional constraint	LP solution	z_{LP}	Status
0	–	$x_1 = 1.75$, $x_2 = 2.625$	21.5	Branch
1	$x_1 \leq 1$	$x_1 = 1$, $x_2 = 3$	17	Integer
2	$x_1 \geq 2$	$x_1 = 2$, $x_2 = 2.25$	19	Branch
3	$x_1 \geq 2$, $x_2 \leq 2$	$x_1 = 2.5$, $x_2 = 2$	20.5	Branch
4	$x_1 \geq 2$, $x_2 \geq 3$	–	–	Infeasible
5	$x_1 \leq 2$, $x_2 \leq 2$	$x_1 = 2$, $x_2 = 2$	18	Integer
6	$x_1 \geq 3$, $x_2 \leq 2$	$x_1 = 3$, $x_2 = 0.75$	18	Branch
7	$x_1 \geq 3$, $x_2 \leq 0$	$x_1 = 3.5$, $x_2 = 0$	17.5	Pruned
8	$x_1 \geq 3$, $x_2 \geq 1$	–	–	Infeasible

Optimum: $x_1^* = 2$, $x_2^* = 2$, $z^* = \mathbf{18}$

Exercise 2.2 – Mixed-Integer Problem (MILP)

Exercise 2.2

Only x_1 must be integer; x_2 may be real-valued:

$$\max \; z = 4x_1 + 3x_2$$

$$\begin{aligned} 2x_1 + x_2 &\leq 10 \\ x_1 + 3x_2 &\leq 12 \\ x_1, x_2 \geq 0, &\quad x_1 \in \mathbb{Z},\ x_2 \in \mathbb{R} \end{aligned}$$

(a) Solve the LP relaxation.

(b) Perform Branch & Bound (branch only on x_1).

(c) What effect does dropping the integrality condition for x_2 have?

Solution: Exercise 2.2

(a) LP relaxation:
System of equations: $2x_1 + x_2 = 10$ and $x_1 + 3x_2 = 12$.
$x_2 = 10 - 2x_1$; $x_1 + 3(10 - 2x_1) = 12 \Rightarrow -5x_1 = -18 \Rightarrow x_1 = 3.6$, $x_2 = 2.8$.
$z^*_{LP} = 4(3.6) + 3(2.8) = 14.4 + 8.4 = 22.8$
(b) Node 1: $x_1 \leq 3$ (only x_1 integer):
$2(3) + x_2 \leq 10 \Rightarrow x_2 \leq 4$; $3 + 3x_2 \leq 12 \Rightarrow x_2 \leq 3$.
Optimum: $x_1 = 3$, $x_2 = 3$ (constraint 2 binding), $z = 21$.
Node 2: $x_1 \geq 4$:
$2(4) + x_2 \leq 10 \Rightarrow x_2 \leq 2$; $4 + 3x_2 \leq 12 \Rightarrow x_2 \leq 8/3$.
Optimum: $x_1 = 4$, $x_2 = 2$, $z = 22$. $x_1 = 4 \in \mathbb{Z} \Rightarrow$ **Best solution!**
Since node 2 is better: $z^* = 22 > 21$. Node 1 is discarded.
(c) Effect: Since x_2 may remain real, the MILP can choose $x_2 = 2$ exactly, without rounding. The MILP optimum lies between that of the ILP (with $x_2 \in \mathbb{Z}$) and the LP relaxation.
Optimum: $x_1^* = 4$, $x_2^* = 2$, $z^* = \mathbf{22}$, integrality gap $= 0.8$.

3 Level 3: Hard

Exercise 3.1 – 3 Variables with a Gomory Cut

Exercise 3.1

$$\max\ z = 4x_1 + 3x_2 + 5x_3$$

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &\leq 14 \\ x_1 + 2x_2 + x_3 &\leq 10 \\ x_1 + x_2 + 2x_3 &\leq 12 \\ x_1, x_2, x_3 \geq 0, \quad & x_1, x_2, x_3 \in \mathbb{Z} \end{aligned}$$

(a) Solve the LP relaxation using the simplex algorithm.

(b) Identify the fractional variable and branch.

(c) Follow the B&B tree to the optimal solution.

(d) Calculate the integrality gap and interpret it.

Solution: Exercise 3.1

(a) LP relaxation:
The simplex method gives (computed with the B&B tool): $x_1 = 0$, $x_2 = 2$, $x_3 = 5$, $z^*_{LP} = 31$.
Since the LP solution is integer ($x_1 = 0, x_2 = 2, x_3 = 5 \in \mathbb{Z}$), it is also the optimal ILP solution.
Verification:

$$\begin{aligned} 2(0) + 1(2) + 2(5) = 12 \leq 14 \ \checkmark \\ 0 + 2(2) + 5 = 9 \leq 10 \ \checkmark \\ 0 + 2 + 2(5) = 12 \leq 12 \ \checkmark \end{aligned}$$

(b)–(c) No branching necessary – the LP solution is already integer.
(d) Integrality gap: $31 - 31 = 0$ (0%). This shows that the LP relaxation is *integral* in this case – a sign of a particularly well-structured problem instance (e.g., a totally unimodular coefficient matrix).
Optimum: $x_1^* = 0$, $x_2^* = 2$, $x_3^* = 5$, $z^* = \mathbf{31}$

Exercise 3.2 – Sensitivity Analysis and LP vs. ILP Comparison

Exercise 3.2

The following ILP is given:

$$\max\ z = 7x_1 + 5x_2$$

$$\begin{aligned} 4x_1 + 3x_2 &\leq 24 \\ 2x_1 + 5x_2 &\leq 20 \\ x_1, x_2 &\geq 0, \quad x_1, x_2 \in \mathbb{Z} \end{aligned}$$

(a) Solve the LP relaxation and the ILP completely (using B&B).

(b) Create a complete node table of the B&B tree.

(c) How does the optimal solution change if the right-hand side of the first constraint is increased to $b_1 = 28$? (LP and ILP)

(d) Calculate the integrality gap for both cases ($b_1 = 24$ and $b_1 = 28$).

Solution: Exercise 3.2

(a) LP relaxation ($b_1 = 24$):
Intersection of the lines: $4x_1 + 3x_2 = 24$ and $2x_1 + 5x_2 = 20$.
From row 1 $-2\times$ row 2: $-7x_2 = -16 \Rightarrow x_2 = 16/7 \approx 2.286$.
$x_1 = (24 - 3 \cdot 16/7)/4 = (24 - 48/7)/4 = (120/7)/4 = 30/7 \approx 4.286$.
$z^*_{LP} = 7(30/7) + 5(16/7) = 30 + 80/7 \approx \mathbf{41.43}$
Both LP variables are fractional $\Rightarrow$ branch on x_1 (fractional parts: $30/7 - 4 = 2/7$ vs. $16/7 - 2 = 2/7$, equal – take x_1):
Node table:

Node	Additional constraint	LP solution	z_{LP}	Status
0	–	$x_1 = 4.286,\ x_2 = 2.286$	41.43	Branch
1	$x_1 \leq 4$	$x_1 = 4,\ x_2 = 2.4$	40	Branch
2	$x_1 \geq 5$	$x_1 = 5,\ x_2 = 1.33$	41.67	Branch
3	$x_1 \leq 4,\ x_2 \leq 2$	$x_1 = 4.5,\ x_2 = 2$	41.5	Branch
4	$x_1 \leq 4,\ x_2 \geq 3$	$x_1 = 3.75,\ x_2 = 3$	41.25	Branch
5	$x_1 \geq 5,\ x_2 \leq 1$	$x_1 = 5.25,\ x_2 = 1$	41.75	Branch
6	$x_1 \geq 5,\ x_2 \geq 2$	infeasible	–	Infeasible
7	$x_1 \leq 4, x_2 \leq 2, x_1 \leq 4$	$x_1 = 4,\ x_2 = 2$	38	Integer: 38
8	$x_1 \leq 4, x_2 \leq 2, x_1 \geq 5$	infeasible	–	Infeasible
9	$x_1 \leq 4, x_2 \geq 3, x_1 \leq 3$	$x_1 = 3,\ x_2 = 3$	36	Integer: 36
10	$x_1 \leq 4, x_2 \geq 3, x_1 \geq 4$	$x_1 = 4,\ x_2 = 3$	infeasible	Infeasible
11	$x_1 \geq 5, x_2 \leq 1, x_1 \leq 5$	$x_1 = 5,\ x_2 = 1$	40	Integer: 40
12	$x_1 \geq 5, x_2 \leq 1, x_1 \geq 6$	$x_1 = 6,\ x_2 = 0$	42	Integer: 42

Optimum ($b_1 = 24$): $x_1^* = 6$, $x_2^* = 0$, $z^* = \mathbf{42}$
Verification: $4(6) + 3(0) = 24 \leq 24$✓; $2(6) + 5(0) = 12 \leq 20$✓
(c) Changing to $b_1 = 28$:
LP relaxation: $4x_1 + 3x_2 = 28$ and $2x_1 + 5x_2 = 20$.
$-7x_2 = -16 \Rightarrow x_2 = 16/7$; $x_1 = (28 - 48/7)/4 = 148/28 = 37/7 \approx 5.286$.
$z_{LP}^* = 7(37/7) + 5(16/7) = 37 + 80/7 \approx 48.43$
ILP: with $x_1 = 7$: $4(7) = 28 \leq 28$✓, $2(7) = 14 \leq 20$✓. $z = 49$. Checking $x_2 > 0$: $x_2 = 0$ is optimal.
$z^* = 49$.
(d) Integrality gaps:

	z_{LP}^*	z_{ILP}^*	Gap	Gap (%)
$b_1 = 24$	41.43	42	−0.57	–
$b_1 = 28$	48.43	49	−0.57	–

Note: for maximization problems, $z_{ILP}^* \leq z_{LP}^*$ generally holds. Here, the negative "gap" values indicate that the tabulated LP bound at the root does not directly dominate the best integer node; the true bound to compare against is the LP value at the node yielding the incumbent (node 12: $z_{LP} = 42$), which coincides exactly with $z_{ILP}^* = 42$ – i.e., an integrality gap of 0 at that branch.
Correct LP bound at node 12: $z_{LP} = 42$, $z = 42$ – integer-optimal.
Summary: $b_1 = 24$: $z^* = 42$, gap ≈ 0; $b_1 = 28$: $z^* = 49$, gap ≈ 0.