Dynamic Input-Output Model

Three Dynamic Leontief Models, Partitioning and Stability

English

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1 Introduction

The dynamic Leontief model extends the static input-output model by the time dimension: investment increases the capital stock, and the capital stock determines future production. The tool contains three variants:

Model	Approach	Requires
Model 1	Static model with a given investment path: $\mathbf{x}(t{+}1) = \mathbf{L}\,[\mathbf{y}{+}\mathbf{B}\,\Delta\mathbf{k}(t)]$	$\mathbf{A}$, $\mathbf{y}$, $\mathbf{B}$ $(n \times n)$
Model 2	Recursive model without partitioning: $\mathbf{x}_{t+1} = \mathbf{B}^{-1}[(\mathbf{E}+\mathbf{B}-\mathbf{A})\mathbf{x}_t - \mathbf{c}_t]$	$\mathbf{A}$, $\mathbf{B}$ $(n \times n)$, $\mathbf{x}_0$, $\mathbf{c}$
Model 3	Recursive model with partitioning into investment goods and consumer goods sectors	$\mathbf{A}$, r, $\mathbf{B}^I$ $(r \times r)$, $\mathbf{x}_0$, $\mathbf{c}$

Here $\mathbf{L} = (\mathbf{E} - \mathbf{A})^{-1}$ is the Leontief inverse and $\mathbf{E}$ the identity matrix. All models require the static model; its result $\mathbf{x}_0 = \mathbf{L}\mathbf{y}$ is the usual start value.

2 Data and Databox

2.1 Loading a Dataset

In the tab *Leontief model – input*, the input method "Databox" is preset. The button **"Load data from databox"** opens a list; the datasets for the dynamic model are at the top, the others collapsed below. After the selection

1. the static model is calculated ($\mathbf{L}$, $\mathbf{x}_0$),
2. the values for all three models are taken over: $\mathbf{B}$, r, $\mathbf{B}^I$, $\mathbf{x}_0$, $\mathbf{c}$, $\mathbf{i}_0$,
3. the display shows matrix $\mathbf{A}$, the vectors per sector, $\mathbf{B}^I$, $\mathbf{B}$ and, for each model, what was taken over or is missing.

In each model, **"Take values from the loaded dataset"** restores the values after own changes. Manual input and CSV upload remain possible.

2.2 Example Datasets

Dataset	Final demand	r	$\mathbf{B}^I$	$\mathbf{B}$
Dynamic 3×3 equilibrium	$\mathbf{y} = \mathbf{c}$	1	1×1	yes
Dynamic 3×3 growth	$\mathbf{y} = \mathbf{c} + \mathbf{i}$	1	1×1	yes
Dynamic 4×4, 6×6, 8×8 equilibrium	$\mathbf{y} = \mathbf{c}$	2, 3, 4	$r \times r$	–
Dynamic 10×10 growth	$\mathbf{y} = \mathbf{c} + \mathbf{i}$	1	$B_I = 5.5$	yes
Example 3×3 to 10×10 (ZIP)	$\mathbf{y} = \mathbf{c} + \mathbf{i}$	1 or 2	$B_I = 5.5$	$B_I\,\mathbf{E}$
Germany 2022 (Destatis)	$\mathbf{y} = \mathbf{c} + \mathbf{i} + \mathbf{v} + \mathbf{e}$	selectable	block of $\mathbf{B}$	illustrative
Germany 2022, investment goods first	reordered	3	block of $\mathbf{B}$	illustrative
ditto, $\mathbf{B}$ scaled	reordered	3	block of $\mathbf{B}$	scaled up

The dynamic examples are fictitious and come from the test data; their start value is $\mathbf{x}_0 = \mathbf{L}\mathbf{y}$. Datasets without a full matrix $\mathbf{B}$ (4×4 to 8×8) are intended for model 3.

Destatis data: they are only of limited suitability for the dynamic models. The capital matrix is illustrative and has small coefficients (diagonal about 0.4–0.75); the simulation therefore turns negative after a few periods. Model 3 treats sectors 1 to r as investment goods sectors – in the Destatis order these are agriculture, raw materials, etc. The tool shows a note on this.

Germany 2022, investment goods first: for model 3, the 20×20 and 15×15 datasets are additionally available in a reordered version: the three sectors with the largest gross fixed capital

formation come first (20×20: construction, research and development, machinery and vehicles, together 69 %; 15×15: groups 7, 11 and 5, together 81 %), $r = 3$ is preset. This makes the choice of r economically meaningful. One variant each has a scaled-up capital matrix (factor about 1.7, mean column sum 3). **These coefficients are illustrative and do not reflect the real data situation.** They only show that larger capital coefficients lengthen the path; the simulation nevertheless stops after a few periods (model 2: 3–5, model 3: 4–7 periods).

2.3 Where $\mathbf{B}^I$ Comes From

For model 3 the tool forms $\mathbf{B}^I$ $(r \times r)$ in this order:

1. own matrix $\mathbf{B}^I$ of the dataset,
2. upper left $r \times r$ block of the capital matrix $\mathbf{B}$,
3. $B_I \cdot \mathbf{E}$ with the scalar capital coefficient B_I.

If the dataset contains no r, r is selected by the user; $\mathbf{B}^I$ is then formed immediately. An r chosen by the user is kept when values are taken over.

3 Given Demand

3.1 C or C + V + E

Models 2 and 3 need the *given* (exogenous) demand $\mathbf{c}_t$. Whatever is not given is treated by the model as investment demand. Options:

- $\mathbf{c} = \mathbf{C} + \mathbf{V} + \mathbf{E}$ (consumption, changes in inventories, exports) – correct for open economies such as the Destatis data; preset there,
- $\mathbf{c} = \mathbf{C}$ (consumption only).

For closed examples ($\mathbf{V} = \mathbf{E} = \mathbf{0}$) both variants are the same.

3.2 Time Path

Variant	Formula	Note
Constant	$\mathbf{c}_t = \mathbf{c}_0$	equilibrium test
Linear	$\mathbf{c}_t = \mathbf{c}_0 + \Delta \cdot t$	Δ per period, e.g. 2
Exponential	$\mathbf{c}_t = \mathbf{c}_0(1+g)^t$	g in %, becomes unstable faster
CSV	own values	

3.3 Suggestion from Y

If the dataset contains no $\mathbf{C}$, "Suggestion from Y" sets $\mathbf{c}_0 = 0.85\,\mathbf{y}$. In model 3, $\mathbf{i}_0 = \mathbf{y} - \mathbf{c}_0$ is set at the same time, so that $\mathbf{y} = \mathbf{c}_0 + \mathbf{i}_0$ is preserved. The suggestion is not needed for the databox examples.

4 Model 1: Given Investment Path

$$\mathbf{x}(t+1) = (\mathbf{E} - \mathbf{A})^{-1}\,[\mathbf{y} + \mathbf{B}\,\Delta\mathbf{k}(t)] \tag{1}$$

$\mathbf{B}$ $(n \times n)$ is the capital coefficient matrix, $\Delta\mathbf{k}(t)$ the given investment vector. Investment scenarios: constant, growing (rate in %), declining (rate in %) or user-defined per period. The model shows which additional production a given capital formation requires; it is stable as long as $\mathbf{L}$ exists.

$\mathbf{B}$ is taken from the databox; without a full matrix, $\mathbf{B} = B_I\,\mathbf{E}$.

5 Model 2: Recursive Model without Partitioning

Production covers intermediate inputs, investment and given demand:

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_t + \mathbf{B}(\mathbf{x}_{t+1} - \mathbf{x}_t) + \mathbf{c}_t \tag{2}$$

Solved for $\mathbf{x}_{t+1}$:

$$\boxed{\mathbf{x}_{t+1} = \mathbf{B}^{-1}\left[(\mathbf{E} + \mathbf{B} - \mathbf{A})\,\mathbf{x}_t - \mathbf{c}_t\right] = \mathbf{x}_t + \mathbf{B}^{-1}[(\mathbf{E} - \mathbf{A})\mathbf{x}_t - \mathbf{c}_t]} \tag{3}$$

The surplus $(\mathbf{E}-\mathbf{A})\mathbf{x}_t - \mathbf{c}_t$ is the investment; via $\mathbf{B}^{-1}$ it increases production in the next period. $\mathbf{B}$ must be invertible.

6 Model 3: Partitioning

6.1 Sectors

With n sectors, sectors 1 to r produce investment goods ($1 \le r < n$), sectors $r+1$ to n consumer goods. The sectors must be sorted in this way. Partitioning of $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} \\ \mathbf{A}^{21} & \mathbf{A}^{22} \end{bmatrix}, \qquad \mathbf{A}^{11} : r \times r, \quad \mathbf{A}^{22} : (n-r) \times (n-r).$$

6.2 Matrices $\mathbf{B}^I$, $\mathbf{B}^d$ and $\mathbf{T}$

$\mathbf{B}^I$ $(r \times r)$ contains the capital coefficients of the investment goods sectors: b^I_{ij} = capital good i that sector j requires per additional unit of output. From this the tool forms

$$\mathbf{B}^d = \begin{bmatrix} \mathbf{B}^I & \mathbf{0} \\ \mathbf{A}^{21} & -(\mathbf{E}^{22} - \mathbf{A}^{22}) \end{bmatrix} \tag{4}$$

The lower $n - r$ rows are the rows of $-(\mathbf{E} - \mathbf{A})$ of the consumer goods sectors. The transition matrix is

$$\mathbf{T} = \mathbf{E} + (\mathbf{B}^d)^{-1}(\mathbf{E} - \mathbf{A}). \tag{5}$$

$\mathbf{B}^I$ and $\mathbf{B}^d$ must be invertible.

6.3 Recursion

$$\boxed{\mathbf{x}_{t+1} = \mathbf{T}\,\mathbf{x}_t - (\mathbf{B}^d)^{-1}\,\mathbf{c}_t} \tag{6}$$

equivalent to $\mathbf{B}^d(\mathbf{x}_{t+1} - \mathbf{x}_t) = (\mathbf{E} - \mathbf{A})\mathbf{x}_t - \mathbf{c}_t$:

- **Investment goods sectors:** the surplus of investment goods determines the increase in capacity via $\mathbf{B}^I$.
- **Consumer goods sectors:** their net production in $t + 1$ exactly covers the given demand, $[(\mathbf{E} - \mathbf{A})\mathbf{x}_{t+1}]^{II} = \mathbf{c}_t^{II}$.

6.4 Start Values

$\mathbf{x}_0 = \mathbf{L}\mathbf{y}$ from the static model, demand $\mathbf{c}_0$ and investment $\mathbf{i}_0$ with $\mathbf{y} = \mathbf{c}_0 + \mathbf{i}_0$. The tool checks this condition (tolerance 1 %) and asks for confirmation in case of deviation. $\mathbf{i}_0$ only serves for the check; the recursion uses $\mathbf{x}_0$ and $\mathbf{c}_t$.

- **Equilibrium** ($\mathbf{y} = \mathbf{c}$, $\mathbf{i}_0 = \mathbf{0}$): with constant $\mathbf{c}$, $\mathbf{x}_t = \mathbf{x}_0$ theoretically remains constant.
- **Growth** ($\mathbf{y} = \mathbf{c} + \mathbf{i}$, $\mathbf{i} > \mathbf{0}$): the investment surplus makes output grow.

7 Stability

7.1 Property of the Model

Computed forwards, the dynamic Leontief model is generally unstable: $\mathbf{T}$ has at least one eigenvalue with modulus greater than 1. Deviations from the equilibrium path – including rounding errors in the start value – grow with the spectral radius $\rho(\mathbf{T}) = \max_i |\lambda_i(\mathbf{T})|$ until a sector has negative output. The simulation then stops and reports the period and the sectors. How many periods are calculated therefore depends on the data.

7.2 Influence of the Capital Coefficients

Larger capital coefficients make the model *more stable*: a surplus is spread over a larger capital requirement and increases production less strongly. Example 3×3, $r = 1$, $\mathbf{x}_0 = \mathbf{L}\mathbf{y}$ with $\mathbf{y} = \mathbf{c}+\mathbf{i}$:

$\mathbf{B}^I$	$\rho(\mathbf{T})$	$x_1(3)$	Behaviour
0.1	8.7	26 612	explosive
0.5	2.54	908	strongly growing
5.5	1.14	276	moderately growing

7.3 Recommendations

- Do not choose capital coefficients too small; values of about 1 to 6 on the diagonal clearly delay the stop.
- Constant or linear demand instead of exponential.
- Take the start value unrounded from the static model.
- For open data use $\mathbf{c} = \mathbf{C} + \mathbf{V} + \mathbf{E}$.

8 Worked Example

8.1 Given

Dataset *Dynamic 3×3 growth*, $r = 1$:

$$\mathbf{A} = \begin{bmatrix} 0.15 & 0.20 & 0.10 \\ 0.25 & 0.10 & 0.15 \\ 0.10 & 0.15 & 0.20 \end{bmatrix}, \quad \mathbf{B}^I = [0.5], \quad \mathbf{c} = \begin{bmatrix} 100 \\ 150 \\ 120 \end{bmatrix}, \quad \mathbf{i}_0 = \begin{bmatrix} 30 \\ 20 \\ 15 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 130 \\ 170 \\ 135 \end{bmatrix}$$

Static model: $\mathbf{x}_0 = \mathbf{L}\mathbf{y} = [254.36;\ 302.42;\ 257.25]^\top$.

8.2 Matrices

$$\mathbf{B}^d = \begin{bmatrix} 0.50 & 0 & 0 \\ 0.25 & -0.90 & 0.15 \\ 0.10 & 0.15 & -0.80 \end{bmatrix}, \qquad (\mathbf{B}^d)^{-1} \approx \begin{bmatrix} 2 & 0 & 0 \\ 0.6165 & -1.1470 & -0.2151 \\ 0.3656 & -0.2151 & -1.2903 \end{bmatrix}$$

$$\mathbf{T} \approx \begin{bmatrix} 2.7000 & -0.4000 & -0.2000 \\ 0.8323 & -0.1233 & -0.0616 \\ 0.4935 & -0.0731 & -0.0366 \end{bmatrix}, \qquad \rho(\mathbf{T}) = 2.54$$

8.3 Simulation (Constant Consumption)

Period	Sector 1	Sector 2	Sector 3
0	254.36	302.42	257.25
1	314.36	294.75	244.56
2	481.97	346.41	275.20
3	907.71	477.65	353.02
4	1 989.16	811.00	550.71

The growth rate of the investment goods sector rises from period to period (factor 1.24; 1.53; 1.88; 2.19) and approaches $\rho(\mathbf{T}) = 2.54$.

Equilibrium for comparison (*Dynamic 3×3 equilibrium*, $\mathbf{y} = \mathbf{c}$): the start value

$$\mathbf{x}_0 = [205.54;\ 261.21;\ 224.67]^\top$$

remains constant over all periods. In the larger examples (4×4 to 8×8, $\rho(\mathbf{T}) \approx$ 7–8), the equilibrium turns negative after 18 to 20 periods because of rounding errors, with a start value rounded to two decimals already after about 7.

Model 2 with the same final demand and the capital matrix $\mathbf{B}$ of the dataset (diagonal 36–44): $\mathbf{x}_1 = [255.14;\ 302.79;\ 257.53]^\top$, $\mathbf{x}_2 = [255.93;\ 303.15;\ 257.81]^\top$ – slow, stable growth.

9 Operation

1. **Input:** "Load data from databox" → choose a dataset. The static model is calculated and the values are taken over.
2. **Model 1:** choose investment scenario and periods → "Start simulation".
3. **Model 2:** check given demand (C or C + V + E) and time path → "Start DynModell2 simulation".
4. **Model 3:** check or choose r, check $\mathbf{B}^I$, demand and $\mathbf{i}_0$; the status list shows what is missing → "Start DynModell3 simulation".
5. **Results:** time series as chart and table; stop on negative output with the period stated.
6. **Export:** Export tab.

10 Validations

- Static model calculated; $(\mathbf{E} - \mathbf{A})$ invertible; warning if a column sum of $\mathbf{A} \geq 1$.
- Model 2: capital matrix $\mathbf{B}$ $(n \times n)$ available and invertible.
- Model 3: $1 \leq r < n$; $\mathbf{B}^I$ is $r \times r$; $\det(\mathbf{B}^I) \neq 0$ and $\mathbf{B}^d$ invertible; $\mathbf{x}_0$ and $\mathbf{c}$ have n values; $\mathbf{y} = \mathbf{c}_0 + \mathbf{i}_0$ (tolerance 1 %, otherwise confirmation request).
- At most 500 periods; stop on negative output.

11 Export

- **Static model:** Leontief inverse, output vector.
- **Model 1:** simulation results, investment path, matrix $\mathbf{B}$.
- **Model 2:** output $\mathbf{x}(t)$, demand $\mathbf{c}(t)$, matrix $\mathbf{B}$, complete export.
- **Model 3:** $\mathbf{B}^I$, $\mathbf{B}^d$, $(\mathbf{B}^d)^{-1}$, $\mathbf{T}$, output $\mathbf{x}(t)$, demand $\mathbf{c}(t)$, final demand $\mathbf{y}$, all simulation data.
- Charts as PNG.

12 Summary

- Three models: given investment path (1), recursion with full capital matrix (2), recursion with partitioning (3).
- Data come from the databox and are taken over for all models.
- Computed forwards, the dynamic Leontief model is unstable; larger capital coefficients and constant or linear demand lengthen the path.
- For open economies, $\mathbf{C} + \mathbf{V} + \mathbf{E}$ belongs to the given demand.

References

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- Ten Raa, T. (2005): *The Economics of Input-Output Analysis*, Cambridge University Press.